PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 27, Number 3, March 1971

REAL LINE BUNDLES ON SPHERES

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ABSTRACT. In a recent paper the author proved a classification theorem for Atiyah-real vector bundles on spaces with free involutions. This result is now applied to the group of Atiyah-real line (i.e., one-dimensional) bundles on spheres, denoted $L_R(S^n)$. It is proved that such bundles are classified by maps into a complex quadric QC^n. Using this classification it is proved that $L_R(S^1)=0$ and that for $n \geqq 3$ the groups are all isomorphic.

1. **Introduction.** Atiyah-real vector bundles and the Grothendieck group $KR(X)$ were defined and studied in [1]. In [2] results were obtained concerning the classification of such bundles by equivariant maps into the complex Grassmann manifold. Specifically, it is shown that when X carries a fixed point free involution there is an isomorphism $\theta: [X, BU-BO]_{\text{eq.}} \to \widetilde{KR}(X)$. Here BO is identified with the fixed point set of BU under the involution induced by complex conjugation. The subscript denotes equivariant maps. Notice that $BU-BO$ has no fixed points.

In this paper the above classification is applied to Atiyah-real line (i.e. one-dimensional) bundles. These bundles form a group under tensor product denoted here by $L_R(X)$. Let $Q(C^n)$ denote the complex quadric with homogenous defining equation $\sum z_i^2=0$, and let $Q=\bigcup_{1\leq n\leq\infty} Q(C^n)$. We then have the following:

PROPOSITION 2. *The inclusion* $S^{n-1}\to S^n$ *defines a natural bijective correspondence* $[S^n, Q]_{\text{eq.}}\to[S^{n-1}, Q]_{\text{eq.}}$ *for* $n\geqq 4$.

COROLLARY. *For* $n\geqq 4$ *there is a natural isomorphism* $L_R(S^n)\to L_R(S^{n-1})$.

PROPOSITION 3. $L_R(S^1)=0$.

Throughout this paper all spaces are compact, Hausdorff. The reader is referred to [1] for the definition and properties of Atiyah-real vector bundles. The sphere S^n is assumed to carry the antipodal

Received by the editors September 19, 1969 and, in revised form, December 22, 1969.

AMS 1970 *subject classifications*. Primary 55F15, 55F50.

Key words and phrases. Atiyah-real vector bundle, conjugate linear isomorphism, equivariant homotopy, line bundle.

[1] This paper forms part of the author's doctoral thesis submitted in June 1968, at the State University of New York, Stony Brook.

involution and the inclusion $S^{n-1}\to S^n$ is onto an equator. Complex conjugation will be denoted $\kappa: C^n \to C^n$.

2. **Classification of real line bundles.** Let $\xi\to X$ be a complex vector bundle, and $\bar{\xi}$ the complex conjugate bundle. Then the natural map $\xi\to\bar{\xi}$ is a conjugate linear isomorphism and an involution $\tau:\xi\to\xi$ defines a conjugate linear isomorphism $\xi_x\to\xi_{\tau(x)}$. An involution is defined on $\bar{\xi}$ by requiring that the following diagram commute:

$$\begin{array}{ccc} \xi_x & \to & \bar{\xi}_x \\ \downarrow & \tau & \downarrow \\ \xi_{\tau(x)} & \to & \bar{\xi}_{\tau(x)} \end{array}$$

The tensor product of two real bundles is again a real line bundle. We require the following extension of a well-known result about complex vector bundles:

LEMMA 1. *Let $\xi\to X$ be a real line bundle. Then $\xi\otimes\bar{\xi}$ is the trivial real line bundle over X.*

PROOF. The standard metric on the classifying bundle is invariant and hence the metric induced in ξ is equivariant in the sense that if $\mu:\xi\otimes\xi\to C$ is this metric, then $\mu(\bar{u},\bar{v})=\kappa(\mu(u,v))$.

Define a vector bundle morphism $f:\xi\otimes\bar{\xi}\to X\times C$ by $f(u_x\otimes v_x) = (x, \mu(u_x, v_x))$. Then f is a surjective morphism of one-dimensional vector bundles and hence is an isomorphism of the underlying complex vector bundles. Since f is equivariant it is an isomorphism of real vector bundles.

COROLLARY. *The real line bundles over X form a group under tensor products.*

Let $L_R(X)$ denote this group. By Proposition II.I of [2] there is a natural bijective correspondence $L_R(X)\leftrightarrow[X, PC^\infty]_{\text{eq.}}$, and if X carries a fixed-point free involution there is, by Proposition II.2 of [2], a natural bijective correspondence $L_R(X)\to[X, PC^\infty - PR^\infty]_{\text{eq.}}$. Let $Q(C^n)$ denote the quadric surface in PC^n defined by $\sum_1^n z_i^2=0$. Then $Q(C^n)$ is invariant under the involution and has no fixed points. It is proved in [3] that PR^n is a deformation retract of $PC^n - QC^n$ in the category of real spaces. Since PC^n/PR^n is homeomorphic to the Thom space of the normal bundle to $Q(C^n)$ in PC^n as real spaces with basepoint (see [3, Corollary 6.4]), by retracting along the fibres we have the following:

LEMMA 2. *$Q(C^n)$ is a deformation retract of $PC^n - PR^n$ as spaces with involution.*

The inclusions $(PC^n - PR^n) \to (PC^{n+1} - PR^{n+1})$ induce inclusions $Q(C^n) \to Q(C^{n+1})$ and commute with the deformation retractions. Hence if we define $Q = \bigcup_{1 \le n \le \infty} Q(C^n)$, Q is a deformation retract of $PC^\infty - PR^\infty$ as spaces with involution.

COROLLARY. *There is a natural bijective correspondence*

$$[X, PC^\infty - PR^\infty]_{\mathrm{eq.}} = [X, Q]_{\mathrm{eq.}}.$$

3. Real line bundles on spheres.

PROPOSITION 1. *The inclusion i: $PC^\infty - PR^\infty \to PC^\infty$ is a weak homotopy equivalence.*

PROOF. Consider the fibration $S^1 \to S^{2n-1} \xrightarrow{p} PC^n$ where S^{2n-1} is viewed as the unit sphere in C^n and p is the projection. Since p is equivariant, $p^{-1}(PR^n) = S^{2n-1} \cap R^n$, where R^n is viewed as the fixed point set of C^n. We then have the restriction

$$S^1 \to S^{2n-1} - S^{n-1} \xrightarrow{p} PC^n - PR^n.$$

If we view S^{2n-1} as the unit sphere in R^{2n} then a point of S^{2n-1} is a pair (X, Y), X, $Y \in R^n$, and $|X|^2 + |Y|^2 = 1$. We can view S^{n-1} as the subset $\{(X, Y) \in S^{2n-1} | Y = 0\}$. Then $\{(X, Y) \in S^{2n-1} | X = 0\}$; also an $n-1$ sphere is a strong deformation retract of $S^{2n-1} - S^{n-1}$. Explicitly, the homotopy may be defined by

$$h_t(X, Y) = (t \cdot X, f(t, Y) \cdot Y)$$

where

$$(f(t, Y))^2 = (1 - t^2 + t^2 |Y|^2)/|Y|^2.$$

It then follows from the exact homotopy sequences of the above fibrations that the inclusion $i: PC^n - PR^n \to PC^n$ defines an isomorphism $i_*: \pi_k(PC^n - PR^n) \to \pi_k(PC^n)$ for $k \leqq n-2$.

Since PC^{n+1} is obtained from PC^n by adjoining a single cell of dimension $2n$, the inclusion $PC^n \to PC^{n+m}$ induces an isomorphism $\pi_k(PC^n) \to \pi_k(PC^{n+m})$ for $k \leqq 2n-1$. From the commutative diagram

$$\begin{array}{ccc} PC^n & \to & PC^{n+m} \\ \uparrow & & \uparrow \\ PC^n - PR^n & \to & PC^{n+m} - PR^{n+m} \end{array}$$

it follows that the inclusion $PC^n - PR^n \to PC^{n+m} - PR^{n+m}$ induces an isomorphism $\pi_k(PC^n - PR^n) \to \pi_k(PC^{n+m} - PR^{n+m})$ for $k \leqq n-2$. Thus by Proposition 4.3 of [4] the inclusions $PC^n \to PC^\infty$ and $PC^n - PR^n \to PC^\infty - PR^\infty$ induce isomorphisms in homotopy for $k \leqq n-2$. Letting

n go to infinity it follows that $i_*:\pi_k(PC^\infty - PR^\infty) \to \pi_k(PC^\infty)$ is an isomorphism for all k. This proves Proposition 1.

PROPOSITION 2. *Let $\Psi: [S^n, Q]_{\text{eq.}} \to [S^{n-1}, Q]_{\text{eq.}}$ be the map induced by the inclusion $S^{n-1} \to S^n$. Then Ψ is injective for $n \geqq 3$ and surjective for $n \geqq 4$.*

PROOF. If $f: S^n \to Q$ is equivariant then so is its restriction to S^{n-1}. Any equivariant homotopy $h: S^n \times I \to Q$ has an eq. (=equivariant) restriction to $S^{n-1} \times I$. Let D_n^+ and D_n^- denote the upper and lower hemispheres of S^n, and τ the antipodal map. The n-skeleton $(D_n^- \times I)_n = D_n^- \times \{0, 1\} \cup S^{n-1} \times I$. There is a map $\alpha: (D_n^- \times I)_n \to Q$ defined by

$$\begin{aligned} \alpha(x, 0) &= f(x), & x &\in D_n^-, \\ \alpha(x, 1) &= g(x), & x &\in D_n^-, \\ \alpha(x, \tau) &= h(x, \tau), & x &\in S^{n-1}. \end{aligned}$$

Clearly these definitions agree on the intersections. Then for $n \geqq 3$, α extends, by Proposition 1, to

$$\begin{aligned} \alpha'(x, \tau) &= h(x, \tau), & x &\in S^{n-1}, \\ \alpha'(x, 0) &= f(x), & x &\in D_n^-, \\ \alpha'(x, 1) &= g(x), & x &\in D_n^-. \end{aligned}$$

We now wish to extend α' to an equivariant homotopy $S^n \times I \to Q$ whose restrictions to $S^n \times \{0\}$ and $S^n \times \{1\}$ are the given maps f and g. We do this using the involution in Q. Define a map $\alpha'': S^n \times I \to Q$ by the following:

$$\begin{aligned} \alpha''(x, \tau) &= \alpha'(x, \tau), & x &\in D_n^-, \\ \alpha''(x, \tau) &= \kappa(\alpha'(\tau(x), \tau)), & x &\in D_n^+. \end{aligned}$$

If $x \in S^{n-1}$, then $\alpha''(x, \tau) = \alpha'(x, \tau) = h(x, \tau)$. Furthermore $\alpha''(x, 0) = \alpha'(x, 0) = f(x)$ for $x \in D_n^-$ and for $x \in D_n^+$ we have $\alpha''(x, 0) = \kappa(\alpha'(\tau(x), 0)) = \kappa(f(\tau(x))) = f(x)$ since f is equivariant. Thus α'' restricted to $S^n \times \{0\}$ agrees with f and by a similar agreement α'' restricted to $S^n \times \{1\}$ agrees with g. This shows that ψ is one-to-one.

Now let $f: S^{n-1} \to Q$ be equivariant. By Proposition 1, f extends to a map $f': D_n^- \to Q$, at least for $n \geqq 4$. Define a map $f'': S^n \to Q$ by the following:

$$\begin{aligned} f''(x) &= f'(x), & x &\in D_n^-, \\ f''(x) &= \kappa(f'(\tau(x))), & x &\in D_n^+. \end{aligned}$$

Then if $x \in D_n^-$,

$$f''(\tau(x)) = \kappa(f'(x)) = \kappa(f''(x))$$

and if $x \in D_n^+$,

$$f''(\tau(x)) = f'(\tau(x)) = \kappa(f''(x))$$

so that f'' is equivariant and agrees with f on S^{n-1}. This implies $\psi[f''] = [f]$ and that for $n \geqq 4$, ψ is onto. This proves Proposition 2. By the naturality of tensor products ψ is in fact a homomorphism, hence we have:

COROLLARY. *For $n \geqq 4$ the inclusion map defines an isomorphism*

$$L_R(S^n) \cong L_R(S^{n-1}).$$

PROPOSITION 3. $[S^1, Q]_{\text{eq.}} = 0$.

PROOF. Let $f, g: S^1 \to Q$ be equivariant. Let $S^0 = \{x_+, x_-\} \subset S^1$. There is a path $\sigma: I \to Q$ with $\sigma(0) = f(x_+)$, $\sigma(1) = g(x_+)$. Then the restrictions of f and g to D_1^+, together with $\sigma(I)$ and $\kappa(\sigma(I))$, define a map $h: S^1 \to Q$ which extends to D_2^+. The extension $h(D_2^+)$ and its conjugate, $\kappa(h(D_2^+))$, define an equivariant homotopy $f \simeq g$ as in the proof of Proposition 2. This completes the proof of Proposition 3.

NOTE. The referee has pointed out that it follows from results of J. Levine, *Spaces with involutions and bundles over P^n*, Amer. J. Math. **85** (1963), that $L_R(S^2)$ is infinite cyclic and $L_R(S^n) = 0$ for $n > 2$.

Levine's equivariant homotopy group $\pi_n^0(X; T) = [S^n, A; X, T]_{\text{eq.}}$ is precisely $L_R(S^n)$ when $X = Q$. With the use of his exact sequence (Theorem 4.3) and the formula on p. 527, the groups can be computed.

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