

AN ASYMPTOTIC PROPERTY OF THE ROOTS OF POLYNOMIALS

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ABSTRACT. It is shown that if the imaginary parts of the roots $\lambda_j(s)$ of a polynomial $P(\lambda, s)$, $s \in R^n$, are unbounded for large $|s|$, then they are in fact unbounded along a one-parameter algebraic curve $s = s(R)$. The result may be used to reduce certain questions about polynomials in several variables to an essentially one-dimensional form; this is illustrated by an application to hyperbolic polynomials.

For a constant-coefficient partial differential equation

$$(1) \quad P(-i\partial/\partial t, -i\partial/\partial x_1, \dots, -i\partial/\partial x_N)u = 0$$

the correctness of the Cauchy problem, in the sense of Petrowsky, is a purely *algebraic question*: do the roots $\lambda_j(s)$ of $P(\lambda, s) = 0$ satisfy

$$(2) \quad |\operatorname{Im} \lambda_j(s)| \leq \text{const}$$

for all s in R^N ? If $N = 1$, the roots can be expanded in a Puiseux series [1]

$$\lambda_j(s) = \sum_{k=-p}^{\infty} a_k s^{q_k}$$

where the exponents q_k tend to $-\infty$ as $k \rightarrow \infty$. Under the assumption that (2) is violated, the function $u_s(t, x) = \exp i(\lambda_j(s)t + sx)$ is a solution of (1) which, either for $t < 0$ or for $t > 0$, grows exponentially in the parameter s , while its values and those of its derivatives at $t = 0$ depend only polynomially on s . Incorrectness of the Cauchy problem is essentially a consequence of this domination of the Cauchy data by the solution. A generalized Puiseux series has been used by Strang [2] to establish necessary conditions for correctness of equations with *variable coefficients*.² This technique does not extend to the case $N > 1$, however, because the Puiseux expansion is then no longer available. If condition (2) should happen to be violated along

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a ray $s = (a_1 r, \dots, a_N r) = ar$, with $a \in R^N$ fixed,

$$\operatorname{Im} \lambda_j(ar) \rightarrow \infty \quad \text{as } r \rightarrow \infty,$$

then correctness could still be established by investigation of a Puiseux expansion in powers of r . Unfortunately, there is an example due to Petrowsky,

$$P(\lambda, s) = \lambda^3 - s_1 \lambda^2 - s_2^2 \lambda + i s_2 \lambda - i s_1 s_2$$

which shows that, although $\operatorname{Im} \lambda_j(ar)$ may be bounded along each ray $s = ar$, it may fail to be bounded uniformly over all rays and may become infinite on more general curves through s -space; in this instance, $\operatorname{Im} \lambda_j(r^2, r) \rightarrow \infty$. The example is, in fact, general. The purpose of this note is to show that if (2) is violated, then violation must occur along an algebraic curve $s = s(R)$ (Theorem 1). Thus there is still a possibility of expanding the λ_j in a Puiseux series in the parameter R , and proceeding with the incorrectness construction as for the case $N = 1$.

The proof of Theorem 1 relies heavily on the Seidenberg-Tarski theorem (see [3] and below), and, as might be expected, can be modified easily to yield information about the manner in which more general algebraic inequalities are violated. This more inclusive statement is given in Theorem 2, followed by a brief sketch of a situation in which it may be of use.

Theorem 1 was conjectured by Professor G. Strang in the course of a study of the correctness of the Cauchy problem [2]; I would like to thank him for numerous helpful suggestions.

2. Let $P(\lambda, s) = \sum_{k=0}^m p_k(s) \lambda^k$, where $p_k(s)$ is a polynomial of degree $m - k$. We define

$$\Lambda(s) = \max_j \operatorname{Im} \lambda_j(s) \quad \text{and} \quad \bar{\Lambda}(r) = \sup_{|s|=r} \Lambda(s).$$

It can be shown (see [3]) that for sufficiently large r , $\bar{\Lambda}(r)$ is an algebraic function: $\bar{\Lambda}(r) = Ar^h + \dots$. We assume that (2) is violated, so that necessarily $h > 0$; let us also suppose that $A > 0$.

THEOREM 1. *Let ϵ satisfying $0 < \epsilon < A$ be given. There exist polynomials G_1, \dots, G_N in two variables, and functions $s_j(R)$, $j = 1, \dots, N$, satisfying*

$$(4) \quad G_j(s_j(R), R) = 0, \quad j = 1, \dots, N,$$

for sufficiently large R , such that

$$(5) \quad \Lambda(s(R)) > (A - \epsilon) |s(R)|^h,$$

for large R . (Here $s(R) = (s_1(R), \dots, s_N(R))$.)

PROOF. Write $\lambda = \mu_1 + i\mu_2$, and $P(\lambda, s) = P_1(\mu_1, \mu_2, s) + iP_2(\mu_1, \mu_2, s)$, with P_1 and P_2 real polynomials. Then by assumption, the following system of (real) polynomial equalities and inequalities will have real solutions for large R :

$$(S_1) \quad \begin{aligned} P_1(\mu_1, \mu_2, s) &= 0, & P_2(\mu_1, \mu_2, s) &= 0, \\ |s|^2 &= R^{2q}, & \mu_2 &> (A - \epsilon)R^p; \end{aligned}$$

we have set $r = R^q$, where p and q are integers with $h = p/q$. According to the Seidenberg-Tarski theorem, there exists a set K_1, \dots, K_d of systems of polynomial equations and inequalities in the two variables s_1 and R , such that a given pair (s_1^0, R^0) may be extended to a solution $s_1^0, s_2, \dots, s_N, \mu_1, \mu_2, R^0$ of (S_1) if and only if (s_1^0, R^0) satisfies at least one of the systems K_j .

Concerning these K_j , we note first that at least one of them must have solutions $R, s_1(R)$ for infinitely many $R, R \rightarrow \infty$, because (S_1) has solutions for large R . In addition, not all the systems K_j may contain an equation. Indeed, if this were the case, then for each R^0 the value of s_1 in any solution s_1, \dots, R^0 of (S_1) would be restricted to a finite set. However, because of the continuous dependence of the roots of $P(\lambda, s) = 0$ on s , if s_1^0, \dots, R^0 is a solution of (S_1) , then a small variation of s_1 may be compensated for by variations of s_2, \dots, μ_2 , without violation of any of the conditions (S_1) . There are, therefore, infinitely many s_1 for which the pair (s_1, R^0) may be extended to a solution of (S_1) . Finally, it is clear that at least one of the K_j which have solutions s_1, R for infinitely many $R, R \rightarrow \infty$, must consist of inequalities only; we take this system to be K_1 .

Suppose K_1 has the form

$$f_i(s_1, R) \geq 0 \quad (\text{or } > 0), \quad i = 1, \dots, e.$$

The solution curves of the equations $f_i(s_1, R) = 0$ divide the (s_1, R) -plane (outside of a bounded region) into finitely many strips; because of the assumption on K_1 , there will be at least one such strip, say T , in the interior of which all f_i are positive. Suppose T is bounded (again, for R large) by the algebraic curves $s_1 = g_1(R), s_2 = g_2(R)$; ultimately, the curve $s_1 = \frac{1}{2}(g_1(R) + g_2(R)) = a_1R^{k_1} + \dots$ will lie in T , and, provided that enough terms are included, so will the partial Puiseux expansion

$$(6) \quad s_1 = h(R) = a_1 R^{k_1} + \dots + a_f R^{k_f}.$$

Put $f_i(h(R), R) = Q_i(R)$. A substitution $R' = R^{1/q'}$, with q' an integer, will convert the Q_i' into expressions involving only powers and reciprocals of R' : $Q_i(R) = Q_i'(R')$. Then multiplication of

$$(f_1(s_1, R'^{q'}) - Q_1'(R'))^2 + \dots + (f_e(s_1, R'^{q'}) - Q_e'(R'))^2$$

by a sufficiently high power of R' will result in a polynomial, $G_1(s_1, R')$, with the following property:

For sufficiently large R' , if $s_1^\circ, R^{\circ'}$ satisfies $G_1(s_1^\circ, R^{\circ'}) = 0$, the pair s_1°, R° (with $R^\circ = R^{\circ'q'}$) may be extended to a solution of (S_1) . Indeed, from $G_1(s_1^\circ, R^{\circ'}) = 0$ follows, for each $i = 1, \dots, e$, that $f_i(s_1^\circ, R^{\circ'q'}) = Q_i'(R^{\circ'})$. Hence $f_i(s_1^\circ, R^\circ) = Q_i(R^\circ) = f_i(h(R^\circ), R^\circ) > 0$, so that s_1°, R° satisfy the system K_1 .

In a similar fashion, one now obtains a polynomial $G_2(s_2, R'')$, where $R'' = (R')^{1/q''}$, with the property:

For sufficiently large R'' , if $(s_2^\circ, R^{\circ''})$ satisfies $G_2(s_2^\circ, R^{\circ''}) = 0$, then $s_2^\circ, R^{\circ''}$ (with $R^{\circ'} = (R^{\circ''})^{1/q''}$) may be extended to a solution of

$$P_1(\mu_1, \mu_2, s) = 0, \quad P_2(\mu_1, \mu_2, s) = 0,$$

(S₂)

$$|s|^2 = (R')^{2qq'}, \quad \mu_2 > (A - \epsilon)(R')^{pq'}, \quad G_1(s_1, R') = 0.$$

It must be noted that (S₂) does indeed have solutions for large R' ; this follows from the defining property of G_1 .

After $N - 1$ steps, one obtains polynomials $G_j(s_j, \bar{R})$, $j = 1, \dots, N - 1$, in s_j and a parameter \bar{R} which is related to r by $r = \bar{R}^{\bar{q}}$, \bar{q} integer, such that the system

$$P_1(\mu_1, \mu_2, s) = 0, \quad P_2(\mu_1, \mu_2, s) = 0,$$

(S_{N-1})

$$|s|^2 = \bar{R}^{2\bar{q}\bar{q}}, \quad \mu_2 > (A - \epsilon)\bar{R}^{p\bar{q}}, \\ G_j(s_j, \bar{R}) = 0, \quad j = 1, \dots, N - 1$$

has solutions for infinitely many \bar{R} , $\bar{R} \rightarrow \infty$.

Now there are only finitely many piecewise algebraic curves $s_1 = s_1(\bar{R}), \dots, s_N = s_N(\bar{R})$ which satisfy the conditions $G_j(s_j(\bar{R}), \bar{R}) = 0, |s|^2 = \bar{R}^{2\bar{q}\bar{q}}$. It follows then that there is a curve $s = s(\bar{R})$, algebraic for sufficiently large \bar{R} , along which we eventually have

$$\mu_2 > (A - \epsilon)\bar{R}^{p\bar{q}}.$$

This proves the theorem.

Trivial modifications in the proof of Theorem 1 serve to establish

THEOREM 2. *Let $s \in R^N$, $t \in R^M$, and let K be a system of real polynomial equations and inequalities in s and t . Suppose further that $\Lambda(s)$ is a function having the property that if t_1 is replaced by $\Lambda(s)$ in each equation or inequality in K , then the resulting system has real solutions t_2, \dots, t_M for all s . If, in addition, $\bar{\Lambda}(r) = \sup_{|s|=r} \Lambda(s)$ is algebraic for large r , and if $\bar{\Lambda}(r) = Ar^h + \dots$ with $h > 0$, then there is an algebraic curve $s = s(R)$, defined by polynomial equations (4), such that the inequality (5) is again satisfied.*

3. In conclusion we outline another situation in which Theorem 2 can be applied. Consider a polynomial $P(\lambda, s) = \sum_{k=0}^m Q_k(\lambda, s)$, expressed as a sum of polynomials Q_k homogeneous of degree k , and suppose that $Q_m(1, 0) = 1$. Let the distinct roots $\lambda = \mu_j(s)$ of $Q_m(\lambda, s) = 0$ be real and of constant multiplicities α_j , $j = 1, \dots, r$. Defining $\partial_j = \lambda - \mu_j(s)$, we write $Q_m(\lambda, s) = \partial_1^{\alpha_1} \dots \partial_r^{\alpha_r}$, or simply $Q_m = \partial^\alpha$, with the multi-index notation $\alpha = (\alpha_1, \dots, \alpha_r)$. Finally, for a general polynomial $R(t)$, $t \in R^M$, define

$$\bar{R}(t) = \left(\sum_{\beta} |(\partial/\partial t)^\beta R(t)|^2 \right)^{1/2},$$

the summation extending over all $\beta = (\beta_1, \dots, \beta_M)$.

It is known from work of Hörmander [4] and Svensson [5], that the roots of $P(\lambda, s)$ will satisfy (2), i.e., P will be hyperbolic, if and only if

$$(7) \quad |Q_k(\lambda, s)| \leq \text{const } \bar{P}(\lambda, s)$$

for all $k = 0, \dots, m$ and $(\lambda, s) \in R^{N+1}$. We now assert that this is equivalent to

$$(8) \quad Q_k \text{ can be written as a linear combination of products } \partial^\beta, \text{ with } \beta \leq \alpha \text{ and } |\beta| = k, \text{ and with coefficients which are bounded functions of } s.$$

PROOF. Let $(\lambda(R), s(R))$ be an algebraic curve. Put $f_j(R) = \lambda(R) - \mu_j(s(R))$, and suppose that f_1, \dots, f_d are the only f_j which tend to zero as $R \rightarrow \infty$. If we now put $\alpha_1 + \dots + \alpha_d = a$ and $\bar{\alpha} = (0, \dots, 0, \alpha_{d+1}, \dots, \alpha_r)$, then it is clear that of all the products $f^\beta = f_1^{\beta_1} \dots f_r^{\beta_r}$, $\beta \leq \alpha$, the one having the most rapid growth with R is $f^{\bar{\alpha}}$. Because

$$\frac{\partial^a}{\partial \lambda^a} Q_m(\lambda, s) = \text{const } \partial^{\bar{\alpha}} + \text{terms involving } \partial_1, \dots, \partial_d,$$

it follows that

$$\frac{\partial^\alpha}{\partial \lambda^\alpha} Q_m(\lambda(R), s(R)) = \text{const } f^{\bar{\alpha}} + o(f^{\bar{\alpha}}), \quad R \rightarrow \infty.$$

This implies that for any β , $\beta \leq \alpha$,

$$|f^\beta| \leq \text{const } \tilde{Q}_m(\lambda(R), s(R)),$$

so that if Q_k has the form prescribed in (8), then the domination inequality (7) is satisfied along each algebraic curve and hence, by Theorem 2, uniformly in λ and s . Condition (8) is therefore sufficient for hyperbolicity.

The necessity of (8) was proved by A. Lax [6] and Yamaguti [7], as was, in fact, the sufficiency; we present our alternate proof partly to illustrate the main theorem, and partly because the same technique can be employed when the roots μ_j have variable multiplicity to produce a description of the lower-order terms of a hyperbolic operator not yet contained in the literature. This extension, as well as applications to equations with variable coefficients, will be discussed in another paper.

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