PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, No. 1, April 1971

ON COLLECTIONS OF SUBSETS CONTAINING NO 4-MEMBER BOOLEAN ALGEBRA

PAUL ERDÖS AND DANIEL KLEITMAN

ABSTRACT. In this paper, upper and lower bounds each of the form $c2^n/n^{1/4}$ are obtained for the maximum possible size of a collection Q of subsets of an n element set satisfying the restriction that no four distinct members A, B, C, D of Q satisfy $A\cup B=C$ and $A\cap B=D$.

The lower bound is obtained by a construction while the upper bound is obtained by applying a somewhat weaker condition on Q which leads easily to a bound. Probably there is an absolute constant c so that

$$\max|Q| = c2^n/n^{1/4} + o(2^n/n^{1/4})$$

but we cannot prove this and have no guess at what the value of c is.

1. **Introduction.** A collection of square free natural numbers which contains no four distinct integers a, b, c, d satisfying l.c.m $(a, b) = c$, g.c.d $(a, b) = d$ naturally corresponds to a collection of sets of prime factors such that no four A, B, C, D satisfy $A\cup B=C$, and $A\cap B=D$. Bounds on the maximal size of the latter kind of collection thus lead to bounds on the former and hence (see Erdös, Sárközi and Szemerédi [1]) to bounds on sums taken over such collections.

In this paper we derive such bounds, which are of the form $c2^n/n^{1/4}$. Analogous results for the corresponding problem when the integers are not required to be square free are indicated. This case corresponds to a collection C of sequences of integers of length n $(S_1, S_2, \cdots, S_n)$ satisfying $S_i\subset(S \max)_i$, such that C contains no four distinct sequences $\{S_i^1\}$, $\{S_i^2\}$, $\{S_i^3\}$, $\{S_i^4\}$ with Max $(S_i^1, S_i^2)=S_i^3$, Min $(S_i^1, S_i^2) = S_i^4$ for all i.

2. **Upper bound.** Let Q be a collection of subsets of an n element set S which satisfies the restriction that no four members A, B, C, D of Q satisfy $A\cup B=C$ and $A\cap B=D$.

Then, if T is any subset of S and W, X, Y, Z are distinct and satisfy $W\subset X\subset T$, $Y\subset Z\subset S-T$ it is not possible that $W\cup Y$, $X\cup Y$, $W\cup Z$ and $X\cup Z$ are all in Q as

$$(X \cup Y) \cap (W \cup Z) = W \cup Y, \quad (X \cup Y) \cup (W \cup Z) = X \cup Z.$$

Received by the editors November 15, 1969.
AMS 1970 *subject classifications*. Primary 05A99.
Key words and phrases. Bounds on collection size, sizes of subset families.

In consequence of this fact, if $X_1, X_2, \cdots, X_k$ and $Y_1, Y_2, \cdots, Y_m$ are all distinct and the X's and Y's are both totally ordered by inclusion with $X_1 \subset X_k \subset T$ and $Y_1 \subset Y_m \subset S-T$, there cannot be two X's and two Y's whose four unions $(X_i \cup Y_j)$ are all in Q. Thus, if we define the zero-one matrix M_{ij} such that $M_{ij}=1$ when $X_i \cup Y_j \in Q$ and $M_{ij}=0$ otherwise, M_{ij} cannot contain a 2×2 submatrix all of whose entries are 1. The total number of entries of M_{ij} which are 1 is the number of members of Q of the form $X_i \cup Y_j$.

The general question of the maximal number $M_{ts}(k, m)$, of $+1$ entries in a $k \times m$ matrix containing no $t \times s$ submatrix all of whose entries are 1 is a well-known problem of Zarankiewicz [**2**] and [**6**]; in the case arising here $t=s=2$, a good bound is a known result due to (Reiman [**3**]); namely

$$(1) \qquad \begin{aligned} M_{22}(k, m) &\leqq \tfrac{1}{2}[k + (k^2 + 4m(m-1)k)^{1/2}], \\ M_{22}(k, m) &\leqq \tfrac{1}{2}[m + (m^2 + 4k(k-1)m)^{1/2}]. \end{aligned}$$

We may apply this result to the problem at hand by partitioning the subsets of T and of $S-T$ into blocks each totally ordered by inclusion. For each pair of blocks F, G of subsets of T and $S-T$ respectively, we may apply the result above to deduce that no more than $M_{22}(f, g)$ members of Q can consist of the union of a member of F with one of G (where f and g are the number of members of F and G respectively).

The maximal size of Q is therefore no more than

$$(2) \qquad \sum_{f,g} p_T(f) p_{S-T}(g) M_{22}(f, g)$$

where $p_T(f)$ and $p_{S-T}(g)$ represent the number of blocks in the partitions of the subsets of T and of $S-T$ having respectively f and g members.

It may be noticed that this bound makes use of a condition somewhat weaker than the original condition on Q. We only here exclude one of four subsets A, B, C, D from Q when $A \cup B = C$ and $A \cap B = D$ if $C-A \subset T$ and $C-B \subset S-T$ or vice versa, for some fixed subset T of S. Moreover the exclusion is only effective if both $C \cap T$ and $D \cap T$ be in the same block in the partition of the subsets of T into blocks each totally ordered by inclusion, and similar remarks hold for $C \cap S-T$ and $D \cap S-T$. In general the restriction will differ for differing choices of T. Families which are invariant under changes of T which maintain its size, are those which contain all members of each of several sizes. For such families the restriction obtained for each T of fixed size are all the same.

We here choose $|T| = [n/2]$. Below we take n divisible by 4 for notational convenience.

It is well known that the subsets of an $n/2$ element set can be partitioned into $\binom{n/2}{n/4}$ blocks each block totally ordered by inclusion. Moreover the number of such blocks of size $2q+1$ can be made equal to $\binom{n/2}{n/4+q} - \binom{n/2}{n/4+q+1}$.

We may therefore set

$$(3) \qquad p_T(2q + 1) = p_{S-T}(2q + 1) = \binom{n/2}{n/4 + q} - \binom{n/2}{n/4 + q + 1}$$

in expression (2) above. It is not possible to partition the subsets into as many blocks that are less regular in size. It can be seen from the expression (1) for $M_{22}(f, g)$ that the size restriction (2) is maximally restrictive when block sizes are maximally unequal, so that this partition will be the most useful for our purposes.

If, instead of subsets of an n element set we were concerned with collections of sequences $(S_1 \cdots S_n)$ of integers satisfying $S_i \subseteq S_i$ max for each i, we could proceed in the same manner. If the indices $l_1 \cdots n$ were divided into two blocks, T and T', sequences restricted to blocks T and T' play the role played above by subsets of T and of $S-T$. If such sequences are divided into totally ordered blocks under the natural orderings ($\{s\} \leqq \{t\}$ if $s_i \leqq t_i$ for all i) the results are identical to those given above, namely the size of Q can be no greater than

$$\sum_{f,g} p_T(f) p_{T'}(g) M_{22}(f, g)$$

where now $p_T(f)$ (and similarly $p_T'(g)$) represents the number of totally ordered blocks of sequences restricted to T having f members in a partition of all such sequences into such blocks. This limitation can be estimated by the same means used for the subset case.

Straightforward manipulation of (2) and (3) yields that an upper bound on the size of our family is

$$\sum_{q=0}^{n/4} 2 \binom{n/2}{n/4 + q} q^{-1/2} \left(\sum_{m=0}^{q} \binom{n/2}{n/4 + m} \right)$$

which quantity behaves as $c_0 2^n n^{-1/4}$.

3. **Lower bound.** We can construct collections Q satisfying the constraint under consideration here as follows. If A, B, C, D are distinct and satisfy $A \cup B = C$, $A \cap B = D$, they must also satisfy the conditions:

$$|A| + |B| = |C| + |D|, \qquad |C| > |A|, \quad |B| > |D|.$$

Thus if we construct a collection of subsets, each of which contains m_i elements for some i, the collection will satisfy our constraint if the m_i's satisfy $m_i+m_j\neq m_k+m_l$ for $m_i\neq m_k$, $m_i\neq m_l$.

It is known [5] that there is a sequence of integers $l\leqq u_1< \cdots <u_k<n^{1/2}$, $k=(1+o(1))n^{1/4}$ satisfying $u_i+u_j\neq u_r+u_s$. Set $m_l=[n/2]+u_l$. Clearly $m_i+m_j\neq m_r+m_s$. Let S satisfy $|S|=n$, and let Q be the collection of all subsets of S having m_l elements for all $l=1, 2, \cdots, k$. Clearly no four elements of Q satisfy $A\cup B=C$, $A\cap B=D$ and further

$$\begin{aligned} |Q| &= \sum_{i=1}^{k}\binom{n}{m_i} > k\binom{n}{m_k} \geqq k\binom{n}{[n/2]+[n^{1/2}]} \\ &\geqq (1+o(1))n^{1/4}\binom{n}{[n/2]+[n^{1/2}]} \\ &> c2^n/n^{1/4} \end{aligned}$$

which establishes our lower bound.

Analogous results may be obtained by use of the same arguments in the case of sequences of integers mentioned earlier in the paper.

References

1. P. Erdös, A. Sárközy and E. Szemerédi, *On the solvability of the equations, $[a_i, a_j]=a_r$ and $(a'_i, a'_j)=a'_r$ in sequences of positive density*, J. Math. Anal. Appl. **15** (1966), 60–64. MR **33** #4035.

2. K. Zarankiewicz, *Problem P* 101, Colloq. Math. **2** (1951), 301. See also: R. K. Guy, *A problem of Zarankiewicz*, Proc. Colloq. Theory of Graphs (Tihany, 1966), Akad. Kiadó, Budapest, 1968, pp. 119–150.

3. I. Reiman, *Über ein Problem von K. Zarankiewicz*, Acta Math. Acad. Sci. Hungar. **9** (1958), 269–273. MR **21** #63.

4. D. J. Kleitman, *On a lemma of Littlewood and Offord on the distribution of certain sums*, Math. Z. **90** (1965), 251–259. MR **32** #2336.

5. P. Erdös and P. Turán, *On a problem of Sidon in additive number theory, and on some related problems*, J. London Math. Soc. **16** (1941), 212–216. See also: P. Erdös, J. London Math. Soc. **19** (1944), 208. MR **3**, 270; MR **7**, 242.

6. R. K. Guy and S. Znám, *A problem of Zarankiewicz*, Recent Progress in Combinatorics, Academic Press, New York, 1969, pp. 237–243.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139