PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, No. 1, April 1971

ON THE ORDER OF THE ERROR FUNCTION OF THE k-FREE INTEGERS

D. SURYANARAYANA AND R. SITARAMACHANDRA RAO

ABSTRACT. Let $\Delta_k(x)$ and $\Delta_k'(x)$ be the error functions in the asymptotic formulae for the number and the sum of k-free integers $\leqq x$. On the assumption of the Riemann hypothesis, we prove the following results by elementary methods:

$$\Delta_k'(x) - x\Delta_k(x) = O(x^{1+3/(4k+1)+\epsilon})$$

and

$$\frac{1}{x}\int_1^x \Delta_k(t)dt = O(x^{3/(4k+1)+\epsilon}),$$

where $\epsilon > 0$.

1. **Introduction.** Let k be a fixed integer $\geqq 2$. A positive integer n is called k-free, if n is not divisible by the kth power of any prime. Let $q_k(n)$ be the characteristic function of the set of k-free integers; that is, $q_k(n) = 1$ or 0 according as n is k-free or not. Let x denote a real variable $\geqq 1$, $[x]$ denote the greatest integer $\leqq x$ and $\{x\}$ denote the fractional part of x; that is, $\{x\} = x - [x]$.

Let $Q_k(x)$ and $Q_k'(x)$ respectively denote the number and the sum of k-free integers $\leqq x$. Let

$$\Delta_k(x) = Q_k(x) - \frac{x}{\zeta(k)}, \tag{1.1}$$

and

$$\Delta_k'(x) = Q_k'(x) - \frac{x^2}{2\zeta(k)}, \tag{1.2}$$

where $\zeta(k)$ is the Riemann zeta function. It is well known that $\Delta_k(x) = O(x^{1/k})$. The best known result has been obtained by A. Walfisz [4], viz., $\Delta_k(x) = O(x^{1/k}\exp(-c_k \log^{3/5} x(\log\log x)^{-1/5}))$, where $c_k > 0$.

In 1911, A. Axer [1] proved on the assumption of the Riemann hypothesis that $\Delta_k(x) = O(x^{2/(2k+1)+\epsilon})$, for every $\epsilon > 0$. The object of the present paper is to establish the following results on the assumption of the Riemann hypothesis:

$$\Delta_k'(x) - x\Delta_k(x) = O(x^{1+3/(4k+1)+\epsilon}), \tag{1.3}$$

and

Received by the editors April 1, 1970.
AMS 1970 *subject classifications*. Primary 10H15.
Key words and phrases. k-free intergers, Riemann hypothesis.

$$\frac{1}{x}\int_1^x \Delta_k(t)dt = O(x^{3/(4k+1)+\epsilon}). \tag{1.4}$$

It is reasonable to conjecture that $\Delta_k(x) = O(x^{3/(4k+1)+\epsilon})$.

The method of the paper is elementary and we assume throughout that the Riemann hypothesis is true. We prepare the necessary background in §2 and give the proofs of (1.3) and (1.4) in §3.

2. **Auxiliary lemmas.** In this section we prove some lemmas which are needed in our present discussion. Throughout the following ϵ denotes any arbitrary positive number.

LEMMA 2.1. (Cf. [2, Theorem 14.25(c), p. 315].) *If $\mu(n)$ denotes the Möbius function, then*

$$M(x) = \sum_{n\leq x} \mu(n) = O(x^{1/2+\epsilon}). \tag{2.1}$$

LEMMA 2.2.

$$N(x) = \sum_{n>x} \frac{\mu(n)}{n^k} = O(x^{-k+1/2+\epsilon}). \tag{2.2}$$

PROOF. Putting $f(n)=1/n^k$ it is not hard to show that $f(n+1)-f(n)=O(1/n^{k+1})$. Therefore by partial summation and (2.1),

$$\begin{aligned}\sum_{n>x} \mu(n)f(n) &= - M(x)f([x]+1) - \sum_{n>x} M(n)(f(n+1)-f(n)) \\ &= O(x^{-k+1/2+\epsilon}) + O\Big(\sum_{n>x} n^{-k-1/2+\epsilon}\Big) \\ &= O(x^{-k+1/2+\epsilon}) + O(x^{-k+1/2+\epsilon}).\end{aligned}$$

Hence Lemma 2.2 follows.

LEMMA 2.3.

$$L(x) = \sum_{n\leq x} \mu(n)n^k = O(x^{k+1/2+\epsilon}). \tag{2.3}$$

PROOF. Putting $g(n)=n^k$ it is clear that $g(n+1)-g(n)=O(n^{k-1})$. Therefore by partial summation and (2.1),

$$\begin{aligned}\sum_{n\leq x} \mu(n)g(n) &= M(x)g([x]) - \sum_{n\leq x-1} M(n)(g(n+1)-g(n)) \\ &= O(x^{k+1/2+\epsilon}) + O\Big(\sum_{n\leq x-1} n^{k-1/2+\epsilon}\Big) \\ &= O(x^{k+1/2+\epsilon}) + O(x^{k+1/2+\epsilon}).\end{aligned}$$

Hence Lemma 2.3 follows.

LEMMA 2.4. *If $z=(x)^{1/k}$ and $\rho=\rho(x)$ is any function of x such that $0<\rho<1$, then*

$$\Delta_k(x) = -\sum_{n\leq\rho z}\mu(n)\left\{\frac{x}{n^k}\right\} + O(\rho^{-k+1/2+\epsilon}z^{1/2+\epsilon}). \tag{2.4}$$

PROOF. We have $q_k(n) = \sum_{d^k|n}\mu(d)$, so that

$$\begin{aligned} Q_k(x) &= \sum_{n\leq x} q_k(n) = \sum_{n\leq x}\sum_{d^k\delta=n}\mu(d) = \sum_{d^k\delta\leq x}\mu(d)\\ &= \sum_{d^k\delta\leq x;\, d\leq\rho z}\mu(d) + \sum_{d^k\delta\leq x;\,\delta\leq\rho^{-k}}\mu(d) - \sum_{d\leq\rho z;\,\delta\leq\rho^{-k}}\mu(d)\\ &= S_1+S_2-S_3, \quad \text{say.}\end{aligned}$$

Now,

$$\begin{aligned} S_1 &= \sum_{d\leq\rho z}\mu(d)\sum_{\delta\leq x/dk}1 = \sum_{d\leq\rho z}\mu(d)\left[\frac{x}{d^k}\right] = x\sum_{d\leq\rho z}\frac{\mu(d)}{d^k} - \sum_{d\leq\rho z}\mu(d)\left\{\frac{x}{d^k}\right\}\\ &= \frac{x}{\zeta(k)} - x\sum_{n>\rho z}\frac{\mu(n)}{n^k} - \sum_{n\leq\rho z}\mu(n)\left\{\frac{x}{n^k}\right\}\\ &= \frac{x}{\zeta(k)} - \sum_{n\leq\rho z}\mu(n)\left\{\frac{x}{n^k}\right\} + O(\rho^{-k+1/2+\epsilon}z^{1/2+\epsilon}),\end{aligned}$$

by Lemma 2.2.

$$\begin{aligned} S_2 &= \sum_{d^k\delta\leq x;\,\delta\leq\rho^{-k}}\mu(d) = \sum_{\delta\leq\rho^{-k}}\sum_{d\leq(x/\delta)^{1/k}}\mu(d) = \sum_{\delta\leq\rho^{-k}}M\left(\left(\frac{x}{\delta}\right)^{1/k}\right)\\ &= O\left(\sum_{\delta\leq\rho^{-k}}\left(\left(\frac{x}{\delta}\right)^{1/k}\right)^{1/2+\epsilon}\right) = O\left(z^{1/2+\epsilon}\sum_{\delta\leq\rho^{-k}}\delta^{-1/2k-\epsilon/k}\right)\\ &= O(z^{1/2+\epsilon}(\rho^{-k})^{1-1/2k-\epsilon/k}) = O(\rho^{-k+1/2+\epsilon}z^{1/2+\epsilon}),\end{aligned}$$

by Lemma 2.1.

$$S_3 = \sum_{d\leq\rho z;\,\delta\leq\rho^{-k}}\mu(d) = M(\rho z)\sum_{\delta\leq\rho^{-k}}1 = O(\rho^{-k+1/2+\epsilon}z^{1/2+\epsilon}),$$

by Lemma 2.1.

Now, substituting the values of S_1, S_2 and S_3 in $Q_k(x)=S_1+S_2-S_3$, we get Lemma 2.4 in virtue of (1.1).

As a consequence of Lemma 2.4, we get an alternative and an elementary proof of A. Axer's result mentioned in the introduction, viz., (2.5) $$\Delta_k(x) = O(x^{2/(2k+1)+\epsilon}).$$

PROOF. By (2.4), we have $\Delta_k(x) = O(\rho z) + O(\rho^{-k+1/2+\epsilon} z^{1/2+\epsilon})$. Now, taking $\rho = x^{-1/k(2k+1)}$, we see that the first O-term is $O(x^{2/(2k+1)})$ and the second O-term is $O(x^{2/(2k+1)+2\epsilon/(2k+1)})$. Hence (2.5) follows.

LEMMA 2.5. *If $z = (x)^{1/k}$ and $\rho = \rho(x)$ is any function of x such that $0 < \rho < 1$, then*

(2.6) $$\Delta_k'(x) = -x \sum_{n \leq \rho z} \mu(n) \left\{\frac{x}{n^k}\right\} + O(\rho^{k+1} z^{k+1}) + O(\rho^{-k+1/2+\epsilon} z^{k+1/2+\epsilon}).$$

PROOF. We have

$$\begin{aligned} Q_k'(x) &= \sum_{n \leq x} Q_k(n) n = \sum_{n \leq x} n \sum_{d^k\delta = n} \mu(d) = \sum_{d^k \delta \leq x} \mu(d) d^k \delta \\ &= \sum_{d^k\delta \leq x; d \leq \rho z} \mu(d) d^k \delta + \sum_{d^k\delta \leq x; \delta \leq \rho^{-k}} \mu(d) d^k \delta - \sum_{d \leq \rho z; \delta \leq \rho^{-k}} \mu(d) d^k \delta \\ &= S_1' + S_2' - S_3', \quad \text{say.} \end{aligned}$$

Now,

$$\begin{aligned} S_1' &= \sum_{d \leq \rho z} \mu(d) d^k \sum_{\delta \leq x/d^k} \delta = \sum_{d \leq \rho z} \mu(d) d^k \frac{1}{2}\left(\left[\frac{x}{d^k}\right]^2 + \left[\frac{x}{d^k}\right]\right) \\ &= \frac{1}{2} \sum_{d \leq \rho z} \mu(d) d^k \left(\left(\frac{x}{d^k} - \left\{\frac{x}{d^k}\right\}\right)^2 + \left(\frac{x}{d^k} - \left\{\frac{x}{d^k}\right\}\right)\right) \\ &= \frac{1}{2} \sum_{d \leq \rho z} \mu(d) d^k \left(\frac{x^2}{d^{2k}} - 2\frac{x}{d^k}\left\{\frac{x}{d^k}\right\} + \left\{\frac{x}{d^k}\right\}^2 + \frac{x}{d^k} - \left\{\frac{x}{d^k}\right\}\right) \\ &= \frac{x^2}{2} \sum_{d \leq \rho z} \frac{\mu(d)}{d^k} - x \sum_{d \leq \rho z} \mu(d) \left\{\frac{x}{d^k}\right\} + \frac{1}{2} \sum_{d \leq \rho z} \mu(d) d^k \left\{\frac{x}{d^k}\right\}^2 \\ &\quad + \frac{x}{2} \sum_{d \leq \rho z} \mu(d) - \frac{1}{2} \sum_{d \leq \rho z} \mu(d) d^k \left\{\frac{x}{d^k}\right\} \\ &= \frac{x^2}{2\zeta(k)} - \frac{x^2}{2} \sum_{n > \rho z} \frac{\mu(n)}{n^k} - x \sum_{n \leq \rho z} \mu(n) \left\{\frac{x}{n^k}\right\} + O(\rho^{k+1} z^{k+1}) \\ &\quad + O(\rho^{1/2+\epsilon} z^{k+1/2+\epsilon}) + O(\rho^{k+1} z^{k+1}) \\ &= \frac{x^2}{2\zeta(k)} - x \sum_{n \leq \rho z} \mu(n) \left\{\frac{x}{n^k}\right\} + O(\rho^{-k+1/2+\epsilon} z^{k+1/2+\epsilon}) \\ &\quad + O(\rho^{k+1} z^{k+1}) + O(\rho^{1/2+\epsilon} z^{k+1/2+\epsilon}), \end{aligned}$$

by Lemma 2.2. Since $\rho^k < 1$, the last O-term in the above is $O(\rho^{-k+1/2+\epsilon} z^{k+1/2+\epsilon})$, so that

$$S_1' = \frac{x^2}{2\zeta(k)} - x \sum_{n \le \rho z} \mu(n) \left\{ \frac{x}{n^k} \right\} + O(\rho^{k+1} z^{k+1}) + O(\rho^{-k+1/2+\epsilon} z^{k+1/2+\epsilon}),$$

$$S_2' = \sum_{\delta \le \rho^{-k}} \delta \sum_{d \le (x/\delta)^{1/k}} \mu(d) d^k = \sum_{\delta \le \rho^{-k}} \delta L\left(\left(\frac{x}{\delta}\right)^{1/k}\right)$$

$$= O\left(\sum_{\delta \le \rho^{-k}} \delta\left(\left(\frac{x}{\delta}\right)^{1/k}\right)^{k+1/2+\epsilon}\right)$$

$$= O\left(z^{k+1/2+\epsilon} \sum_{\delta \le \rho^{-k}} \delta^{-1/2k-\epsilon/k}\right) = O(z^{k+1/2+\epsilon}(\rho^{-k})^{1-1/2k-\epsilon/k})$$

$$= O(\rho^{-k+1/2+\epsilon} z^{k+1/2+\epsilon}),$$

by Lemma 2.3.

$$S_3' = \sum_{\delta \le \rho^{-k}} \delta \sum_{d \le \rho z} \mu(d) d^k = \sum_{\delta \le \rho^{-k}} \delta L(\rho z) = O(\rho^{-2k} (\rho z)^{k+1/2+\epsilon})$$

$$= O(\rho^{-k+1/2+\epsilon} z^{k+1/2+\epsilon}),$$

by Lemma 2.3.

Now, substituting the values of S_1', S_2' and S_3' in $Q_k'(x) = S_1' + S_2' - S_3'$, we get Lemma 2.5 in virtue of (1.2).

3. **Proofs of (1.3) and (1.4).** We have by Lemma 2.4 and Lemma 2.5,

$$\Delta_k'(x) - x\Delta_k(x) = O(\rho^{k+1} z^{k+1}) + O(\rho^{-k+1/2+\epsilon} z^{k+1/2+\epsilon}), \tag{3.1}$$

for every function $\rho = \rho(x)$ such that $0 < \rho < 1$. Now, choosing $\rho(x) = x^{-1/k(4k+1)} = z^{-1/(4k+1)}$, we see that the first O-term in (3.1) is $O(x^{1+3/(4k+1)})$ and the second O-term is $O(x^{1+3/(4k+1)+4\epsilon/(4k+1)})$.

Hence (1.3) follows.

We have by partial summation and (1.1),

$$Q_k'(x) = \sum_{n \le x} q_k(n) n = xQ_k(x) - \sum_{n \le x-1} Q_k(n)((n+1) - n)$$

$$= xQ_k(x) - \int_1^x Q_k(t)dt$$

$$= \frac{x^2}{\zeta(k)} + x\Delta_k(x) - \int_1^x \left(\frac{t}{\zeta(k)} + \Delta_k(t)\right) dt$$

$$= \frac{x^2}{\zeta(k)} + x\Delta_k(x) - \frac{x^2}{2\zeta(k)} + O(1) - \int_1^x \Delta_k(t)dt$$

$$= \frac{x^2}{2\zeta(k)} + x\Delta_k(x) - \int_1^x \Delta_k(t)dt + O(1),$$

so that by (1.2),

$$\Delta_k'(x) - x\Delta_k(x) = -\int_1^x \Delta_k(t)dt + O(1).$$

Hence (1.4) follows in virtue of (1.3). Thus (1.3) and (1.4) are proved.

Finally, we remark that A. M. Vaidya [3] considered the case of square-free numbers and has attempted to give an analytical proof of the stronger result, viz., $\Delta_2'(x) - x\Delta_2(x) = O(x^{1+1/4+\epsilon})$. But there is a mistake in his proof. He argued on p. 200, lines 16 and 17 of his paper that

$$\text{“}O\left(x^{1/4+\epsilon}\cdot \lim_{\delta\to 0}\left(\int_{-T}^{-\delta} + \int_{\delta}^{T}\right)|\, t\,|^{3/8-k-1+\epsilon}\right) = O(x^{1/4+\epsilon}),$$

since the integrals converge.” The mistake is that the integrals do not converge.

References

1. A. Axer, *Über einige Grenzertsatze*, S.-B. Akad. Wiss. Wien IIa **120** (1911), 1253–1298.

2. E. C. Titchmarsh, *The theory of the Riemann Zeta function*, 2nd ed., Clarendon Press, Oxford, 1951. MR **13**, 741.

3. A. M. Vaidya, *On the order of the error function of the square-free numbers*, Proc. Nat. Inst. Sci. India A **32** (1966), 196–201.

4. A. Val'fiš (Walfisz), *Weylsche Exponentialsummen in der neueren Zahlentheorie*, Mathematische Forschungsberichte, 15, VEB Deutscher Verlag, Berlin, 1963. MR **36** #3737.

Andhra University, Waltair, India