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A NEW CHARACTERIZATION OF DEDEKIND DOMAINS

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ABSTRACT. In this note it is shown that a Noetherian ring R is a Dedekind domain if every maximal ideal M of R satisfies the cancellation law: if A and B are nonzero ideals of R and $MA = MB$, then $A = B$.

Let R be a Noetherian domain (commutative with 1). And let S be the semigroup of ideals of R under multiplication. It is well known that R is a Dedekind Domain if, and only if, every element $A \in S$ satisfies the cancellation law: if B, $C \in S$ and $A \neq 0$, then $AB = AC$ implies $B = C$. Since a Dedekind domain has the property that every ideal is a product of primes, however, it is natural to ask if the assumption that every ideal is cancellable is necessary. In this note we show that a Noetherian ring is a Dedekind domain if every maximal ideal is cancellable.

For an extensive bibliography on Dedekind domains we refer the reader to [1].

The main tool used in the following is the theorem, due to Samuel [2], that if Q is an ideal primary for the maximal ideal of a local ring R, then for sufficiently large values of n, the length of R/Q^n is a polynomial in n of degree equal to the rank of M. We denote this polynomial by $p_Q(x)$.

We begin with the following:

LEMMA. *Let R be a local ring in which the maximal ideal M satisfies the cancellation law. Then either $M = 0$ or M has rank 1.*

PROOF. Since M satisfies the cancellation law, either $M = 0$ or $0:M = 0$. In the second case, set $M = (a_1, \cdots, a_d)$ and let $p(x)$ be the polynomial $p_M(x+1) - p_M(x)$. Then for sufficiently large values of n, $p(n)$ is the length of the R-module M^n/M^{n+1}, which is also the number of elements in a minimal base for M^n. Now, for all $n \geqq 1$, $M^{nd+n} = M^{nd}(a_1^n, \cdots, a_d^n)$, so, by cancellation, $M^n = (a_1^n, \cdots, a_d^n)$. Hence $p(n) \leqq d$ for all sufficiently large n. Since $0:M = 0$, it follows that $p(x)$ has degree 0, and therefore that $p_M(x)$ has degree 1. Hence M has rank 1. Q.E.D.

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THEOREM. *Let R be a Noetherian ring such that every maximal ideal satisfies the cancellation law. Then R is a Dedekind Domain.*

PROOF. Assume that R is not a field. It suffices to show that for every maximal ideal M, R_M is a regular local ring of altitude 1. To do this, fix M and set $\overline{R}=R_M$. We adopt the notation that for any ideal A of R, $\overline{A}=AR_M$. Then $\overline{A}\overline{M}:\overline{M}=(AM:M)R_M=\overline{A}$, so the maximal ideal $\overline{M}$ of the local ring $\overline{R}$ is cancellable. Since $\overline{M}\neq 0$, $\overline{M}$ has rank 1 by the Lemma. Clearly, $\overline{M}$ is not a prime of 0 in $\overline{R}$, so there exists an element $a\in\overline{R}$ such that $a\in\overline{M}$, $a\notin\overline{M}^2$, and a is not an element of any prime of 0 (see, for example, [3, p. 406]). Then the ideal (a) is primary for $\overline{M}$, so there exists an integer k such that $\overline{M}^k\not\subseteq(a)$ and $\overline{M}^{k+1}\subseteq(a)$ (where $\overline{M}^k=\overline{R}$ if $k=0$). Hence $\overline{M}^{k+1}=\overline{M}^{k+1}\cap(a)=(\overline{M}^{k+1}:(a))(a)$; and therefore either $\overline{M}^{k+1}\subseteq\overline{M}(a)$ or $\overline{M}^{k+1}=(a)$. However, if $\overline{M}^{k+1}\subseteq\overline{M}(a)$, then $\overline{M}(a)=\overline{M}(\overline{M}^k+(a))$ and $(a)=\overline{M}^k+(a)$, which contradicts the choice of k. Hence $\overline{M}^{k+1}=(a)$, so by the choice of a, $k=0$ and $\overline{M}$ is principal. Since $\overline{M}$ is not a prime of 0 in $\overline{R}$, this completes the proof.

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