OVERRINGS OF PRINCIPAL IDEAL DOMAINS

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Abstract. All rings between a (right and left) principal ideal domain \( R \) and its skewfield \( Q(R) \) of quotients are quotient rings of \( R \) with respect to Ore-systems in \( R \).

In \([4]\) R. Gilmer and J. Ohm investigate commutative integral domains with the \( QR \)-property, i.e. with the property that every ring between the ring \( R \) and its quotient field \( Q(R) \) is a ring of quotients of \( R \) with respect to some multiplicatively closed system in \( R \). All Dedekind domains with torsion class group have the \( QR \)-property.

In this note we describe all overrings (i.e. rings between \( R \) and its skewfield \( Q(R) \) of quotients) of a principal right and left ideal domain \( R \), as left quotient rings with respect to appropriate Ore-systems in \( R \). Contrary to the commutative case, we show that the left quotient ring with respect to an irreducible element need not be local. Also, a counterexample shows that for a principal left ideal domain which is not a principal right ideal domain, overrings exist which are not left quotient rings with respect to some Ore-system. See \([1]\) for similar results.

Let \( R \) be a principal left and right ideal domain. Then \( R \) is a unique factorization domain, in which every nonunit element \( 0 \neq a \) can be written as a product of irreducible elements:

\[ a = p_1 \cdots p_n, \quad p_i \text{ irreducible for all } i. \]

If \( a = q_1 \cdots q_m \) is another irreducible factorization of \( a \), then \( n = m \) and there exists a permutation \( \pi \) of \( \{1, \ldots, n\} \) such that \( R/Rp_i \) is isomorphic to \( R/Rq_{\pi(i)} \) as an \( R \)-left module for every \( i \) (\([5, p. \text{34}]\)). We say an element \( a \) is similar to \( b \) in \( R \), if \( R/Ra \) is isomorphic to \( R/Rb \) as a left-\( R \)-module. It is well known, \([3, p. \text{316}]\), that \( a \) and \( b \) are similar if and only if there exists an element \( c \) in \( R \) with \( Ra + Rc = R \) and \( Rac \cap Rb = Rbc \). Similarity is an equivalence relation and left-right symmetric.

For each \( a \neq 0 \) in \( R \) and every irreducible element \( p \) in \( R \) we define a nonnegative integer \( h_p(a) = n \), if \( n \) is the number of irreducible factors similar to \( p \) in any irreducible factorization of \( a \). It follows...
that \( h_p(a) = 0 \) for all \( p \) if and only if \( a \) is a unit in \( R \). For any set \( \Delta \) of irreducible elements in \( R \) we define the set \( S_\Delta = \{ a \in R \mid h_p(a) = 0 \text{ for all } p \in \Delta \} \).

We repeat the following definition: A multiplicatively closed set \( M \) of regular elements of \( R \) is called a (left-) Ore-system if for elements \( m \) in \( M \) and \( r \) in \( R \) there exist elements \( m_1 \) in \( M \) and \( r_1 \) in \( R \) such that \( r_1m = mr \). If \( M \) is an Ore-system in \( R \), then there exists the ring of quotients of \( R \) with respect to \( M \) ([6, p. 7]). With this notation we obtain the following result:

**Theorem.** Let \( R \) be a left and right principal ideal domain. Then any set \( S_\Delta \) is an Ore-system and every ring between \( R \) and its skewfield of quotients \( Q(R) \) can be obtained as a ring of quotients of \( R \) with respect to some \( S_\Delta \).

**Proof.** First we show that every \( S_\Delta \) is an Ore-system. That \( S_\Delta \) is multiplicatively closed follows from the unique factorization theorem stated in the beginning. It remains to show that for \( s \in S_\Delta \) and \( r \) in \( R \) elements \( s_1 \) in \( S_\Delta \) and \( r_1 \) in \( R \) exist with

\[
s_1r = r_1s.
\]

Let \( Rs + Rr = Rd, \ s = s'd, \ r = r'd \), and it follows

\[
Rs' + Rr' = R \quad \text{for } s' \text{ in } S_\Delta \text{ and } r' \text{ in } R.
\]

Further, \( Rs' \cap Rr' = Rs_1r' \) and \( s_1 \) is a product of irreducible factors similar to the irreducible factors of \( s' \) and is therefore contained in \( S_\Delta \) ([2, p. 51]). It follows that \( s_1r' = r_1s' \) or \( s_1r = r_1s \) for \( s_1 \) in \( S_\Delta \) and \( r_1 \) in \( R \). This proves that \( S_\Delta \) is an Ore-system and there exists an over-ring \( S_\Delta^{-1}R = \{ s^{-1}a, \ s \in S, \ a \in R \} \) of \( R \).

Now let \( T \) be any ring between \( R \) and \( Q(R) \), \( a^{-1}b \) an element in \( T \). We may assume \( bR + aR = R \), and from \( bx + ay = 1 \), for \( x, y \) in \( R \), it follows that \( a^{-1} \) belongs to \( T \). Therefore, the inverse of every irreducible factor of \( a \) is contained in \( T \). It remains to show that for \( p_1 \) similar to the irreducible element \( p \) in \( R \) with \( p^{-1} \) in \( T \) we have that \( p_1^{-1} \) is contained in \( T \).

Since \( p_1 \) is similar to \( p \), there exists \( c \) in \( R \) with \( Rp \cap Rc = Rp_1c \) and \( Rp + Rc = R \). The element \( cp^{-1} \) is contained in \( T \) and can be written in the form \( cp^{-1} = a^{-1}b \) for \( a \neq 0 \), \( b \) in \( R \). But then \( ac = bp \) in \( Rp \cap Rc \) and this means \( ac = rp_1c \) or \( a = rp_1 \) for some element \( r \) in \( R \). From this it follows that \( p_1^{-1} \) is contained in \( T \). This proves that \( T = S_\Delta^{-1}R \) where \( \Delta \) is the set of those irreducible \( q \) in \( R \) such that \( q^{-1} \) is not contained in \( T \).

**Remark 1.** Unlike the commutative case, \( S_\Delta^{-1}R \) is in general not a
local ring for $\Delta$ consisting of just one irreducible element $p$. ($S_p^{-1}R$ is local if and only if $h_p(a+b) \geq \min(h_p(a), h_p(b))$ [2, p. 57].) For example, consider the irreducible element $p=x+t$ in the principal left and right ideal domain $R=\mathbb{Q}(t)[x, \sigma]=\{\sum_{i=0}^{m} f_i(t)x^i, f_i(t) \in \mathbb{Q}(t)\}$, where $\mathbb{Q}(t)$ is the field of rational functions in one variable over the field of rationals $\mathbb{Q}$ and $\sigma$ is the automorphism of $\mathbb{Q}(t)$ defined by $t^\sigma = t+1$. Addition in $R$ is defined componentwise and multiplication by $xf(t) = f^\sigma(t)x$.

Now it is easy to see that $x+g(t)^{-1}g(t+1)(t+1)$ for every element $0 \neq g(t) in \mathbb{Q}(t)$, is similar to the element $x+t$ in $R$. From this we conclude that for example $S_p^{-1}Rp$ and $S_p^{-1}Rp_1$, with $p_1=x+t^{-1}(t+1)^2$, are two different maximal left ideals of $S_p^{-1}R$ for $p=x+t$.

Remark 2. The theorem just proved can not be extended to left principal ideal domains with maximum condition on right principal ideals, even though the rings $S^{-1}R$, as defined above, exist in this case too. Consider $\mathbb{Q}(t)[x, t]=R$ as in the previous remark, but $\sigma$ is now the monomorphism from $\mathbb{Q}(t)$ into $\mathbb{Q}(t)$ defined by $t^\sigma = t^2$. $R$ is a principal left ideal domain with maximum condition on principal right ideals. Therefore the skewfield $Q(R)$ of quotients, $Q(R) = \{a^{-1}b, 0 \neq a, b in R\}$, exists, but the subring $T$ of $Q(R)$ generated by $R$ and $x^{-1}tx$, $T=R[x^{-1}tx]$, is not a ring of quotients of $R$ with respect to some Ore-system of $R$. To prove this last statement we observe first that $R[x^{-1}tx] = \{ax^{-1}dxb+c for a, b, c, d in R\}$ since $(x^{-1}tx)^2 = t$. If we assume $T=S^{-1}R$ for some Ore-system $S$ of $R$, it follows that there exists an element $f(x)$ in $R$ of degree greater than 0, such that $f(x)^{-1}$ is in $T$. It is easy to prove that $f(x)$ has to be equal to $ux$ for a unit $u$ in $R$ and it would follow that $x^{-1}$ is contained in $T$. But this is not possible since from $x^{-1} = ax^{-1}dxb+c, a, b, c, d in R$, it follows that $1 = a_1dxb+xc$ for some $a_1 in R$, leading to a contradiction.

References