OVERRINGS OF PRINCIPAL IDEAL DOMAINS

H. H. BRUNGS

Abstract. All rings between a (right and left) principal ideal domain $R$ and its skewfield $Q(R)$ of quotients are quotient rings of $R$ with respect to Ore-systems in $R$.

In [4] R. Gilmer and J. Ohm investigate commutative integral domains with the $QR$-property, i.e. with the property that every ring between the ring $R$ and its quotient field $Q(R)$ is a ring of quotients of $R$ with respect to some multiplicatively closed system in $R$. All Dedekind domains with torsion class group have the $QR$-property.

In this note we describe all overrings (i.e. rings between $R$ and its skewfield $Q(R)$ of quotients) of a principal right and left ideal domain $R$, as left quotient rings with respect to appropriate Ore-systems in $R$. Contrary to the commutative case, we show that the left quotient ring with respect to an irreducible element need not be local. Also, a counterexample shows that for a principal left ideal domain which is not a principal right ideal domain, overrings exist which are not left quotient rings with respect to some Ore-system. See [1] for similar results.

Let $R$ be a principal left and right ideal domain. Then $R$ is a unique factorization domain, in which every nonunit element $0 \neq a$ can be written as a product of irreducible elements:

$$a = p_1 \cdot \cdots \cdot p_n, \quad p_i \text{ irreducible for all } i.$$

If $a = q_1 \cdot \cdots \cdot q_m$ is another irreducible factorization of $a$, then $n = m$ and there exists a permutation $\pi$ of $\{1, \cdots, n\}$ such that $R/Rp_i$ is isomorphic to $R/Rq_{\pi(i)}$ as an $R$-left module for every $i$ ([5, p. 34]). We say an element $a$ is similar to $b$ in $R$, if $R/Ra$ is isomorphic to $R/Rb$ as a left-$R$ module. It is well known, [3, p. 316], that $a$ and $b$ are similar if and only if there exists an element $c$ in $R$ with $Ra + Rc = R$ and $Ra \cap Rc = Rbc$. Similarity is an equivalence relation and left-right symmetric.

For each $a \neq 0$ in $R$ and every irreducible element $p$ in $R$ we define a nonnegative integer $h_p(a) = n$, if $n$ is the number of irreducible factors similar to $p$ in any irreducible factorization of $a$. It follows

Received by the editors May 4, 1970.

AMS 1970 subject classifications. Primary 16A04; Secondary 16A08.

Key words and phrases. Overrings, principal ideal domains, local rings.

1 This research was supported in part by the NRC of Canada, under Grant no. A-7254.
that \( h_p(a) = 0 \) for all \( p \) if and only if \( a \) is a unit in \( R \). For any set \( \Delta \) of irreducible elements in \( R \) we define the set \( S_\Delta = \{ a \in R \mid h_p(a) = 0 \text{ for all } p \in \Delta \} \).

We repeat the following definition: A multiplicatively closed set \( M \) of regular elements of \( R \) is called a (left-) Ore-system if for elements \( m \) in \( M \) and \( r \) in \( R \) there exist elements \( m_1 \) in \( M \) and \( r_1 \) in \( R \) such that \( r_1 m = m_1 r \). If \( M \) is an Ore-system in \( R \), then there exists the ring of quotients of \( R \) with respect to \( M \) ([6, p. 7]). With this notation we obtain the following result:

**Theorem.** Let \( R \) be a left and right principal ideal domain. Then any set \( S_\Delta \) is an Ore-system and every ring between \( R \) and its skewfield of quotients \( Q(R) \) can be obtained as a ring of quotients of \( R \) with respect to some \( S_\Delta \).

**Proof.** First we show that every \( S_\Delta \) is an Ore-system. That \( S_\Delta \) is multiplicatively closed follows from the unique factorization theorem stated in the beginning. It remains to show that for \( s \in S_\Delta \) and \( r \) in \( R \) elements \( s_1 \) in \( S_\Delta \) and \( r_1 \) in \( R \) exist with

\[ s_1 r = r_1 s. \]

Let \( Rs + Rr = Rd, s = s'd, r = r'd, \) and it follows

\[ Rs' + Rr' = R \quad \text{for } s' \text{ in } S_\Delta \text{ and } r' \text{ in } R. \]

Further, \( Rs' \cap Rr' = R s' r' \) and \( s_1 \) is a product of irreducible factors similar to the irreducible factors of \( s' \) and is therefore contained in \( S_\Delta \) ([2, p. 51]). It follows that \( s_1 r' = r_1 s' \) or \( s_1 r = r_1 s \) for \( s_1 \) in \( S_\Delta \) and \( r_1 \) in \( R \). This proves that \( S_\Delta \) is an Ore-system and there exists an over-ring \( S_\Delta^{-1} R = \{ s^{-1} a, s \in S, a \in R \} \) of \( R \).

Now let \( T \) be any ring between \( R \) and \( Q(R) \), \( a^{-1} b \) an element in \( T \). We may assume \( b R + a R = R \), and from \( bx + ay = 1 \), for \( x, y \) in \( R \), it follows that \( a^{-1} \) belongs to \( T \). Therefore, the inverse of every irreducible factor of \( a \) is contained in \( T \). It remains to show that for \( p_1 \) similar to the irreducible element \( p \) in \( R \) with \( p^{-1} \) in \( T \) we have that \( p_1^{-1} \) is contained in \( T \).

Since \( p_1 \) is similar to \( p \), there exists \( c \) in \( R \) with \( R p \cap R c = R p_1 c \) and \( R p + R c = R \). The element \( c p^{-1} \) is contained in \( T \) and can be written in the form \( c p^{-1} = a^{-1} b \) for \( a \neq 0, b \) in \( R \). But then \( a c = b p \) in \( R p \cap R c \) and this means \( a c = r p_1 c \) or \( a = r p_1 \) for some element \( r \) in \( R \). From this it follows that \( p_1^{-1} \) is contained in \( T \). This proves that \( T = S_\Delta^{-1} R \) where \( \Delta \) is the set of those irreducible \( q \) in \( R \) such that \( q^{-1} \) is not contained in \( T \).

**Remark 1.** Unlike the commutative case, \( S_p^{-1} R \) is in general not a
local ring for \(\Delta\) consisting of just one irreducible element \(p\). \((S_p^{-1}R\) is local if and only if \(h_p(a+b) \geq \min(h_p(a), h_p(b))\) \cite[p. 57]{[2]}. For example, consider the irreducible element \(p = x+t\) in the principal left and right ideal domain \(R = \mathbb{Q}(t)[x, \sigma] = \{\sum_{i=0}^{\infty} f_i(t) x^i, f_i(t) \in \mathbb{Q}(t)\}\), where \(\mathbb{Q}(t)\) is the field of rational functions in one variable over the field of rationals \(\mathbb{Q}\) and \(\sigma\) is the automorphism of \(\mathbb{Q}(t)\) defined by \(t^2 = t+1\). Addition in \(R\) is defined componentwise and multiplication by \(xf(t) = f'(t)x\).

Now it is easy to see that \(x + g(t)^{-1} g(t+1)(t+1)\) for every element \(0 \neq g(t)\) in \(\mathbb{Q}(t)\), is similar to the element \(x+t\) in \(R\). From this we conclude that for example \(S_p^{-1}Rp\) and \(S_p^{-1}R\pi_1\), with \(\pi_1 = x+t-1(t+1)^2\), are two different maximal left ideals of \(S_p^{-1}R\) for \(p = x+t\).

**Remark 2.** The theorem just proved can not be extended to left principal ideal domains with maximum condition on right principal ideals, even though the rings \(S^{-1}R\), as defined above, exist in this case too. Consider \(\mathbb{Q}(t)[x, t] = R\) as in the previous remark, but \(\tau\) is now the monomorphism from \(\mathbb{Q}(t)\) into \(\mathbb{Q}(t)\) defined by \(t^2 = t^2\). \(R\) is a principal left ideal domain with maximum condition on principal right ideals. Therefore the skewfield \(Q(R)\) of quotients, \(Q(R) = \{a^{-1}b, 0 \neq a, b \in R\}\), exists, but the subring \(T\) of \(Q(R)\) generated by \(R\) and \(x^{-1}tx, T = R[x^{-1}tx]\), is not a ring of quotients of \(R\) with respect to some Ore-system of \(R\). To prove this last statement we observe first that \(R[x^{-1}tx] = \{ax^{-1}dxb+c \text{ for } a, b, c, d \in R\}\) since \((x^{-1}tx)^2 = t\). If we assume \(T = S^{-1}R\) for some Ore-system \(S\) of \(R\), it follows that there exists an element \(f(x)\) in \(R\) of degree greater than 0, such that \(f(x)^{-1}\) is in \(T\). It is easy to prove that \(f(x)\) has to be equal to \(ux\) for a unit \(u\) in \(R\) and it would follow that \(x^{-1}\) is contained in \(T\). But this is not possible since from \(x^{-1} = ax^{-1}dxb+c, a, b, c, d \in R\), it follows that \(1 = a_1dxb + xc\) for some \(a_1\) in \(R\), leading to a contradiction.

**References**


University of Alberta, Edmonton, Alberta, Canada