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DECOMPOSITION OF FUNCTION-LATTICES

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ABSTRACT. We give a simple direct proof of the theorem (due to Kaplansky-Blair-Burrill) that the lattice $C(X, K)$ of all continuous functions defined on the topological space X with values in the chain K can be decomposed iff X contains an open-and-closed subset.

For any topological space X, let $C(X, K)$ denote the lattice of all K-valued continuous functions defined on X, where K is any non-singleton totally ordered set with the order topology. Clearly, if A is any open-and-closed subspace of X, then $C(X, K)$ is lattice isomorphic to the direct product $C(A, K)\times C(X\backslash A, K)$. Improving a technique of Kaplansky [2], Blair and Burrill [1] have shown that a converse holds. We give a simple alternative proof of this result which, in contrast to the proofs of Kaplansky and Blair-Burrill, avoids use of the axiom of choice. For this observation and several other suggestions for improving the presentation we are grateful to the referee.

A sublattice $L\subseteq C(X, K)$ is *adequate* provided that, for each $x\in X$, there are functions $f, g\in L$ such that $f(x) \neq g(x)$.

THEOREM. *If an adequate sublattice L of $C(X, K)$ is lattice isomorphic to the direct product $L_1\times L_2$ of lattices L_1 and L_2, then there is an open-and-closed subset $A\subseteq X$ such that L_1 is lattice isomorphic to $\{f| A : f\in L\}$ and L_2 is lattice isomorphic to $\{f| (X\backslash A) : f\in L\}$.*

We first establish a

LEMMA. *Let L_1 and L_2 be lattices and K be a totally ordered set. If $\alpha: L_1\times L_2\rightarrow K$ is a lattice homomorphism, then one of the following holds:*
(1) *For any $k, k'\in L_2$, $\alpha(l, k)=\alpha(l, k')$ for any $l\in L_1$.*
(2) *For any $l, l'\in L_1$, $\alpha(l, k)=\alpha(l', k)$ for any $k\in L_2$.*
Moreover, if α is not constant, then precisely one of these holds.

PROOF. Note that (1) is equivalent to:
(1′) For any $k, k'\in L_2$, $\alpha(l_0, k)=\alpha(l_0, k')$ for some $l_0\in L_1$.
This follows from the observation that

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$$\alpha(l, k) = \alpha(((l_0, k) \wedge (l, k \vee k')) \vee (l, k \wedge k'))$$
$$= \alpha(((l_0, k') \wedge (l, k \vee k')) \vee (l, k \wedge k')) = \alpha(l, k').$$

Similarly, (2) is equivalent to:

(2′) For any $l, l' \in L_1$, $\alpha(l, k_0) = \alpha(l', k_0)$ for some $k_0 \in L_2$.

Now, assume that condition (2) fails. Then, there exist $l, l' \in L_1$ such that $\alpha(l, k) \neq \alpha(l', k)$ for any $k \in L_2$. To show that condition (1) holds, consider any $k, k' \in L_2$. Then,

$$\alpha(l \wedge l', k \vee k') \vee \alpha(l \vee l', k \wedge k') = \alpha(l \vee l', k \vee k').$$

Since K is totally ordered and $\alpha(l \wedge l', k \vee k') = \alpha(l \vee l', k \vee k')$ implies that $\alpha(l, k \vee k') = \alpha(l', k \vee k')$ (a contradiction), we conclude that $\alpha(l \vee l', k \wedge k') = \alpha(l \vee l', k \vee k')$. Hence, $\alpha(l \vee l', k) = \alpha(l \vee l', k')$ so that conditions (1′) and (1) hold. Evidently, both conditions hold iff α is constant.

Proof of the Theorem. Let L be an adequate sublattice of $C(X, K)$ and $\psi: L_1 \times L_2 \to L$ be a lattice isomorphism. For each $x \in X$, the lattice homomorphism $\varphi_x: L \to K$, defined by $\varphi_x(f) = f(x)$, is not constant. From the preceding lemma $\varphi_x \circ \psi: L_1 \times L_2 \to K$ satisfies one, and only one, of the conditions (1) and (2). Define $A = \{x \in X : \varphi_x \circ \psi$ satisfies condition (1)$\}$. It follows easily that A and $X \backslash A$ are disjoint closed sets. Finally, define $\theta: L_1 \to \{f|A : f \in L\}$ by $\theta(l) = f|A$, where $f = \psi(l, k_0)$ for some $k_0 \in L_2$. It follows directly that θ is a lattice isomorphism. Similarly, one considers $X \backslash A$ so that the proof of the theorem is complete.

Remarks. An easy corollary is that a topological space X is connected iff, for any totally ordered set K, there is no adequate sublattice $L \subseteq C(X, K)$ which is lattice isomorphic to the direct product $L_1 \times L_2$ of two lattices L_1 and L_2, neither of which is a singleton. Hence, a topological space X is connected iff every extension of X is connected (where an extension of X is any topological space that contains X as a dense subspace).

References

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