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MINIMAL IMMERSIONS OF 2-SPHERES IN S^4

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ABSTRACT. The classification of isolated singularities of minimal varieties leads to the study of minimal manifolds in the n-sphere. The object of this paper is to show that a minimal 2-sphere in S^4 with trivial normal bundle is the standard 2-sphere.

In [1] Almgren showed that the only minimal 2-sphere in S^3 is the standard S^2. For minimal spheres in S^4 an additional assumption is needed to rule out counterexamples. The Veronese surface, a minimal imbedding of the projective plane in S^4 is such an example. For details see [3].

Some of the ideas in this paper are related to a paper by H. Hopf [2] in which he proves that a 2-sphere immersed with constant mean curvature in E^3 is the standard S^2. This connection is further emphasized by Theorem I, a generalization of Hopf's theorem.

2. **Definitions and results.** Throughout the paper, [4, Chapter VII] is used as a standard reference. For convenience we recall a few definitions here:

The mean curvature H, a cross section of the normal bundle, is defined to be the trace of the second fundamental form. A surface is called minimal in S^{n-1} if the mean curvature vector H is parallel to the normal vector of S^{n-1} in E^n. For technical reasons it is convenient to consider the minimal surfaces in S^{n-1} as surfaces in E^n. Of course the surfaces are no longer minimal, but the mean curvature vector H is still parallel; i.e., the covariant derivative of H in the normal bundle is zero. However, for spheres nothing new is added as the following theorem shows.

THEOREM I. *If $I:S^2\to E^n$ is an immersion of S^2 into E^n with parallel mean curvature vector of length one, then the image $I(S^2)$ is a minimal surface in S^{n-1}.*

The next theorem is the main result of this paper.

THEOREM II. *If $I:S^2\to S^4$ is an immersion of S^2 into S^4 as a minimal*

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surface with trivial normal bundle, then $I(S^2)$ is the standard totally geodesic 2-sphere in S^4.

Here we might add that, as the referee points out, the hypothesis "with trivial normal bundle" is equivalent to "is regularly homotopic to the standard immersion" by the Smale-Hirsch theory of immersions.

3. **Proofs.** The main step in the proof of Theorem II consists in showing that the curvature of the normal bundle does not change sign. This step, as well as the proof of Theorem I, depends on a system of partial differential equations for sections of the normal bundle. These equations reduce to the Cauchy-Riemann equations in the codimension-one case.

The local computations are made in an isothermal coordinate system. The Codazzi equations, together with the assumption that H is parallel, yield the following system of equations:

$$(*) \qquad \nabla_Y u - \nabla_X v = 0, \qquad \nabla_X u + \nabla_Y v = 0$$

for the sections u and v of the normal bundle which will be defined shortly.

The system (*) is obtained as in [**2**, p. 240]. If α denotes the second fundamental form of the immersion, and if $X=\partial/\partial x$, $Y=\partial/\partial y$ denote the coordinate vector fields of an isothermal coordinate system, then the Codazzi equations, [**4**, p. 25], in terms of the sections $L=\alpha(X,X)$, $M=\alpha(X,Y)$, and $N=\alpha(Y,Y)$ may be written as follows:

$$\nabla_Y L - \nabla_X M = \frac{YE}{2E}(L+N) = (YE)H,$$

$$\nabla_Y M - \nabla_X N = -\frac{XE}{2E}(L+N) = -(XE)H,$$

where $E(dx^2+dy^2)$ is the first fundamental form; and XE denotes the derivative of E in the direction X. In order to give the above system the desired form, we differentiate $2EH=L+N$, the defining equation for H and we obtain $(YE)H=-E\nabla_Y H+\frac{1}{2}(\nabla_Y L+\nabla_Y N)$. Using this equality, the first Codazzi equation takes the following form: $\nabla_Y((L-N)/2)-\nabla_X M=-E\nabla_Y H=0$, where $\nabla_Y H$ is zero because H is parallel. With the substitution $u=(L-N)/2$ and $v=M$ the above equation is identical to the first equation of (*). Similarly, the second equation of (*) is obtained.

For the proof of Theorem I, one considers the components u_1 and v_1 of u and v respectively in the direction of H. Since H is parallel, the system (*) also holds for the components u_1 and v_1; i.e., u_1+iv_1 is anti-holomorphic in terms of the isothermal parameters. Now we use the maximum modulus theorem for anti-holomorphic functions to show that u_1+iv_1 is identically zero. To achieve the latter, the complex plane is used as an isothermal coordinate system for the sphere minus a point. Since $|u_1+iv_1|$ is bounded by $2kE$ where k is an upper bound for the principal curvatures, and since the limit of E as $|x+iy|$ tends to infinity is zero, we have $u_1+iv_1\equiv 0$. This in turn implies that the principal curvatures in the H-direction are both equal to $\|H\|=1$. Now we consider the map $I-H: S^2\to \boldsymbol{R}^n$ that sends a point $x\in S^2$ into the point $I(x)-H(x)\in\boldsymbol{R}^n$. Since both principal curvatures are equal to one, the derivative of $I-H$ is identically zero; and therefore, $I-H$ is identically equal to a constant $c\in\boldsymbol{R}^n$. Thus, $I=c+H$, i.e., $I(S^2)$ is a surface in the unit sphere with center $c\in\boldsymbol{R}$. In fact, $I(S^2)$ is a minimal surface in S^{n-1} because the mean curvature vector H is perpendicular to S^{n-1}.

For the proof of Theorem II, the system (*) will be used again. Since the components of u and v in the H-direction are zero and H is parallel, one can consider the system (*) to be a system of equations for sections in the orthogonal complement to H in the normal bundle. In the first step it will be shown that $\|u\|^2-\|v\|^2$ as well as $\langle u, v\rangle$ are identically zero. The system (*) implies

$$\nabla_X\nabla_X u + \nabla_Y\nabla_Y u = \nabla_Y\nabla_X v - \nabla_X\nabla_Y v = R(Y, X)v,$$
$$\nabla_X\nabla_X v + \nabla_Y\nabla_Y v = -\nabla_Y\nabla_X u + \nabla_X\nabla_Y u = -R(Y, X)u,$$

where $R(Y, X)$ denotes the curvature tensor of the normal bundle. Applying the Laplace operator to $\|u\|^2-\|v\|^2$ results in the following:

$$\begin{aligned} E\Delta(\|u\|^2 - \|v\|^2) &= 2(\|\nabla_X u\|^2 + \|\nabla_Y u\|^2 - \|\nabla_X v\|^2 - \|\nabla_Y v\|^2) \\ &\quad + 2(\langle R(Y, X)v, u\rangle + \langle R(Y, X)u, v\rangle) \\ &= 0, \end{aligned}$$

where the first parenthesis is zero because of (*), while the second parenthesis is zero because $\langle R(Y, X)u, v\rangle$ is skew symmetric in u and v. Again, we consider the isothermal coordinate system introduced in the proof of Theorem I. Since both u and v tend to zero for large $|x+iy|$, it follows from the maximum principle for harmonic functions that $\|u\|^2-\|v\|^2\equiv 0$. In a like manner, $\langle u, v\rangle\equiv 0$ is proved.

The system (*) is further used to show that the sections u and v are either identically zero or have isolated zeros only. Let p be a point where both u and v vanish. For convenience, we introduce a canonical coordinate system with center p for the normal bundle; i.e., we identify the restriction of the normal bundle to a neighborhood U with the Cartesian product $U \times \boldsymbol{R}^2$ by means of a cross section obtained by parallel translation of a frame over p along geodesic rays. Finally, we consider the nonvanishing terms u_k and v_k of lowest order in the Taylor formula for u and v respectively. The system (*) implies that $u_k + iv_k$ is anti-holomorphic. Therefore, the sections u and v have isolated zeros only. The alternative $u = v \equiv 0$ will not be considered again since in this case the proof is completed.

With these preparations out of the way, we compute the curvature $\langle R(X, Y)u, v\rangle$ of the normal bundle of the minimal 2-sphere in S^4. By the formulas obtained in §2.3 of [**5**] we have $\langle R(X, Y)u, v\rangle = \langle [A^u, A^v]X, Y\rangle$, where A^u denotes the linear transformation associated to the second fundamental form in the direction of u. If unit vectors in the direction of u and v are used as a basis for the normal bundle, we obtain

$$[A^u, A^v] = \left[\begin{pmatrix} \|u\| & 0 \\ 0 & -\|u\| \end{pmatrix}, \begin{pmatrix} 0 & \|v\| \\ \|v\| & 0 \end{pmatrix}\right]$$
$$= \begin{pmatrix} 0 & 2\|u\|\,\|v\| \\ -2\|u\|\,\|v\| & 0 \end{pmatrix}.$$

Therefore, $\langle R(X, Y)u, v\rangle = -2\|u\|\,\|v\| = -2\|u\|^2$. It follows that the curvature of the normal bundle is equal to $\pm 2\, u^2/E$, the sign depending on the orientation of u and v. The integral of this curvature over the 2-sphere is equal to the Euler number of the normal bundle. This number is zero because the normal bundle is assumed to be trivial. Since the curvature is either zero in isolated points only, or is identically zero, it cannot change sign and therefore must be identically zero. The immediate implication is that the minimal 2-sphere in S^4 is in fact the standard geodesic S^2.

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