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A CHARACTERIZATION OF PUNCTURED OPEN 3-CELLS

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ABSTRACT. A proof is given using standard methods of the topology of three-dimensional manifolds of the following characterization of punctured cubes: A connected, open 3-manifold M is topological E^3 with k points removed if and only if every polyhedral simple closed curve in M lies in a topological cube in M and the rank of $\pi_2(M)$ is k. An application is given.

1. **Introduction.** Bing has proved [1, Theorem 1] that a compact, connected 3-manifold M is topologically S^3 if each simple closed curve in M lies in a topological cube in M. He proceeds to show [1, Theorem 2] that a bounded, connected, open subset of E^3 is topologically E^3 if the boundary of U is connected and each polyhedral simple closed curve in U lies in a topological cube in U. We propose to improve the latter result so that it more closely resembles Bing's characterization of S^3.

DEFINITION. A manifold M will be called an open manifold if M is noncompact and has empty boundary.

THEOREM 1. *A connected, open 3-manifold M is topologically E^3 if and only if every polyhedral simple closed curve in M lies in a topological cube in M and $\pi_2(M)$ is trivial.*

Considering this theorem as the initial step in an induction proof produces

THEOREM 2. *A connected, open 3-manifold M is topologically E^3 with k points removed (a punctured cube) if and only if every polyhedral simple closed curve in M lies in a topological cube in M and the rank of $\pi_2(M)$ is k.*

Another application of Theorem 1 results in

THEOREM 3. *A connected, open, irreducible 3-manifold M is topologically E^3 if each polyhedral simple closed curve in M lies in a homologically trivial complex in M.*

2. **Proofs of the theorems.** Recall from [1] that if M is a connected

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3-manifold in which each polyhedral simple closed curve J lies in a topological cube in M, then

(a) J lies interior to a polyhedral cube in M,

(b) each polyhedral finite graph in M lies interior to a polyhedral cube in M,

(c) each compact subset of the 2-skeleton of any triangulation of M lies interior to a punctured cube (a cube with finitely many points removed).

(c) is not explicitly stated but is proved in the course of proving Lemma 4 in [**1**].

The proof of our Theorem 1 relies on

LEMMA 1. *If M is an open connected 3-manifold in which each polyhedral simple closed curve lies in a topological cube then every compact set in M lies interior to a polyhedral punctured cube in M.*

PROOF. Let C be compact in M. Then C lies in a compact, connected, simplicial neighborhood L in M. Choose a regular neighborhood N of L in M. Since N is a compact, connected 3-manifold with boundary, N has a 2-dimensional spine X. Now X lies interior to a polyhedral, punctured cube in M, and because N collapses to X, N also lies interior to a polyhedral punctured cube in M. But $N \supset C$.

The following definition and theorem are due to McMillan [**3**].

DEFINITION. A 3-manifold M is a W-space if M is open, contractible, and every compact set in M has a neighborhood which embeds in E^3.

THEOREM. *Let M be a W-space. Then $M = \bigcup_{i=1}^{\infty} C_i$ where each C_i is a cube with handles and $C_i \subset \operatorname{Int} C_{i+1}$.*

PROOF OF THEOREM 1. First, because $\pi_1(M)$ and $\pi_2(M)$ are trivial, $\pi_3(M) \approx H_3(M) = 0$ because M is noncompact. Thus M is contractible, and, by Lemma 1, every compact set in M has a neighborhood which embeds in E^3. Therefore, M is a W-space and so must be the monotone union $\bigcup_{i=1}^{\infty} C_i$ of polyhedral cubes with handles. Next we will prove that each compact set K in M lies interior to a polyhedral cube in M. Because K is compact, K lies interior to some C_i. Now C_i collapses to a polyhedral finite graph Y_i, and since such graphs lie interior to polyhedral cubes in M, C_i and hence K lie interior to a polyhedral cube in M. It follows that M is the monotone union of 3-cells, which, according to Brown [**2**], means that M is topologically E^3.

The proofs of Theorems 2 and 3 use the following

LEMMA 2. *If M is a 1-connected 3-manifold with $\partial M=\emptyset$ and Σ is a 2-sphere in M, then Σ separates $M-\Sigma$ into exactly two components.*

PROOF. H_c^* is Alexander-Spanier cohomology with compact support. Now $0=\pi_1(M)\approx H_1(M)\approx H_c^2(M)$ by Poincaré duality so $0\rightarrow H_c^2(\Sigma)\xrightarrow{\delta} H_c^3(M-\Sigma)\rightarrow H_c^3(M)\rightarrow 0$ is exact. But $H_c^2(\Sigma)\approx Z$ and $H_c^3(M)\approx Z$ so $Z\oplus Z\approx H_c^3(M-\Sigma)\approx H_0(M-\Sigma)$.

Our last lemma is

LEMMA 3. *Let M be a connected, open 3-manifold in which every polyhedral simple closed curve lies in a topological cube, and let Σ be a polyhedral 2-sphere in M such that $M-\Sigma=U\cup V$ where U and V are disjoint, open, connected sets.*

Then the open manifold M_1 obtained by attaching a ball D^3 to the closure of U along Σ also has the property that every polyhedral simple closed curve in M_1 lies in a topological cube in M_1.

PROOF. Let S be a polyhedral simple closed curve in M_1. Since $(S\cap U)\cup\Sigma$ is 2-dimensional, it lies interior to a polyhedral punctured cube C_1 in M. By Lemma 2, Σ separates C_1 into components $U\cap C_1$ and $V\cap C_1$, each of which is a punctured cube. To see this, repair the punctures in C_1 and split the resulting cube along Σ and then puncture each component as C_1 was punctured. Now attach $(U\cap C_1)\cup\Sigma$ to the ball D^3 along Σ to obtain a punctured cube in M_1 which contains S. Thus S lies in a cube in M_1.

PROOF OF THEOREM 2. We proceed by induction on the rank of $\pi_2(M)=k$. For $k=0$, the orientability of M guarantees that $\pi_2(M)\approx H_2(M)$ is torsion-free so that $\pi_2(M)=0$.

If $k>0$, the Whitehead sphere theorem [4] allows us to find a polyhedral 2-sphere Σ in M which represents a generator of $\pi_2(M)$. By Lemma 2, $M-\Sigma$ is the union of two components, U and V. Since M is 1-connected, so are U and V. As a consequence, $\pi_2(M)\approx H_2(M)$, $\pi_2(U)\approx H_2(U)$, and $\pi_2(V)\approx H_2(V)$. From the Mayer-Vietoris sequence of the pair $(\overline{U}, \overline{V})$, the exactness of the sequence $0\rightarrow H_2(\Sigma)\rightarrow H_2(\overline{U})\oplus H_2(\overline{V})\rightarrow H_2(M)\rightarrow 0$ results. Thus if rank $[\pi_2(U)]=n$ and rank $[\pi_2(V)]=m$, then $n+m=k+1$. By Lemma 3, attaching balls to $\overline{U}$ and $\overline{V}$ along Σ we obtain two open 3-manifolds satisfying the conditions of the induction hypothesis. Thus they are E^3 with, respectively, $n-1$ and $m-1$ points removed. By detaching the balls and reconstructing M we clearly get E^3 with $(n-1)+(m-1)+1=k$ points removed.

PROOF OF THEOREM 3. It suffices to prove that each polyhedral simple closed curve in M lies in a topological cube in M. To this end, let S be such a curve in M. Choose a homologically trivial complex K in M containing S and let N be a regular neighborhood of K in M. Now N is a compact 3-manifold with boundary ∂N and N has the homotopy type of K. Thus $0=H^1(N;\, Z_2)\approx H_2(N,\, \partial N;\, Z_2)$ by Poincaré duality and so from the exact sequence for the pair $(N, \partial N)$,

$$\cdots \to H_2(N,\, \partial N;\, Z_2) \to H_1(\partial N;\, Z_2) \to H_1(N;\, Z_2) \to \cdots,$$

we get $H_1(\partial N;\, Z_2)=0$. It follows that ∂N is a 2-sphere, so must bound a 3-cell. Because the closure of each component of $M-N$ is noncompact, N must be that 3-cell. Then S lies in a cube.

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