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NO TOPOLOGIES CHARACTERIZE DIFFERENTIABILITY AS CONTINUITY

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ABSTRACT. Do there exist topologies $\mathcal{U}$ and $\mathcal{V}$ for the set R of real numbers such that a function f from R to R is smooth in some specified sense (e.g., differentiable, C^n, or C^∞) with respect to the usual structure of the real line if and only if f is continuous from $\mathcal{U}$ to $\mathcal{V}$? We show that the answer is no.

First assume $\mathcal{U}=\mathcal{V}$: we establish that if, with respect to a single given topology on R, (a) all functions of the form $x \rightsquigarrow px+q$ ($p, q \in R$) are continuous, and (b) some nonzero function h from R to R which vanishes on the negative numbers is continuous, then the function $k: x \rightsquigarrow |x|$ is also continuous.

To prove this we may evidently assume the given topology is not indiscrete. Consequently we can, using (a), find an open set U to include any chosen nonzero w and to exclude 0. We choose w in $h[R]$: then $h^{-1}[U]$ is a nonvoid open set containing only nonnegative numbers. Whenever $x \neq 0$, therefore, $|x|$ has, by (a), a neighborhood consisting of nonnegative numbers. For any such neighborhood N, however, $k^{-1}[N]=N \cup (-N)$, which, by (a), is a neighborhood of x. It follows that k is continuous at x. To prove that k is continuous at 0, note that if M is a neighborhood of 0 so is $M^*=M\cap(-M)$, and that $k^{-1}[M^*]=M^*$.

The following lemma now shows that there was no loss of generality in the assumption that $\mathcal{U}=\mathcal{V}$.

LEMMA. *Let X be a set, and let $C(\mathcal{U}, \mathcal{V})$—defined whenever $\mathcal{U}$, $\mathcal{V}$ are topologies on X—denote the class of $(\mathcal{U}, \mathcal{V})$-continuous maps. Suppose $C(\mathcal{U}, \mathcal{V})$ is a class F which* (i) *is closed under composition and* (ii) *includes the identity map. Then $F=C(\mathfrak{I}, \mathfrak{I})$ for some $\mathfrak{I}$.*

PROOF. Let $\mathcal{S}$ be the topology generated on X by the collection

$$\{f^{-1}[V]: f \in F, V \in \mathcal{V}\}.$$

Then $F \subset C(\mathcal{S}, \mathcal{V}) \subset C(\mathcal{U}, \mathcal{V})$; hence $F=C(\mathcal{S}, \mathcal{V})$. Let $\mathfrak{I}$ be the topology

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$$\{T : f \in F \Rightarrow f^{-1}[T] \in \mathcal{S}\}.$$

Then $F \subset C(\mathcal{S}, \mathfrak{T}) \subset C(\mathcal{S}, \mathcal{V})$; hence $F = C(\mathcal{S}, \mathfrak{T})$. By (i), each generator of $\mathcal{S}$ belongs to $\mathfrak{T}$: consequently $\mathcal{S} \subset \mathfrak{T}$. By (ii), therefore, $\mathcal{S} = \mathfrak{T}$.

The following question remains open: Does there exist a topology on R with respect to which a function is continuous if and only if it is *analytic*?

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