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A COMPARISON THEOREM

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ABSTRACT. In this paper the authors consider a pair of differential equations $y_1''+p_1(x)y_1=0$, $y_2''+p_2(x)y_2=0$, where $p_i(x)$ are positive and continuous, and where solutions $y_1(x)$ and $y_2(x)$ have common consecutive zeros at $x=a$ and $x=b$. They show that if the curves $y=p_1(x)$ and $y=p_2(x)$ have a single intersection (possibly a closed subinterval) and if $p_1(a)>p_2(a)$, $p_2(b)>p_1(b)$, the first conjugate point of $a+\epsilon$ ($\epsilon>0$ and small) for the second equation precedes that of the first.

Consider the differential equations

$$y_1'' + p_1(x)y_1 = 0, \tag{1}$$

$$y_2'' + p_2(x)y_2 = 0, \tag{2}$$

where the functions $p_i(x)$ are positive and continuous on an interval $I: [a, b+\delta]$ $(\delta>0)$. If solutions $y_1(x)$ and $y_2(x)$ of equations (1) and (2), respectively, have common consecutive zeros at $x=a$ and $x=b$, and if $p_1(a)>p_2(a)$, it follows from the Sturm comparison theorem that the curves $y=p_1(x)$ and $y=p_2(x)$ must intersect. In recent years a number of papers have been concerned with differential equations of the above type when these curves have a single point of intersection. Notable among these are Fink [2], [3], Eliason [4] and [5].

In the present paper we assume that the curves $y=p_i(x)$ intersect once on the interval (a, b), but the intersection may be either a point or a closed subinterval of (a, b). We have the following result.

THEOREM. *If the curves $y=p_1(x)$ and $y=p_2(x)$ have the properties described above, if equations* (1) *and* (2) *have solutions $y_1(x)$ and $y_2(x)$, respectively, for which $x=a$ and $x=b$ are consecutive zeros, and if*

$$p_1(a) > p_2(a), \qquad p_2(b) > p_1(b), \tag{3}$$

then, for $\epsilon>0$ and sufficiently small, the first conjugate point of $x=a+\epsilon$ for equation (2) *precedes the first conjugate point of $x=a+\epsilon$ for equation* (1).

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To begin, we recall a result due to one of the present writers [6].

LEMMA. *If* c *is a conjugate point of* x_0 *with respect to the equation*[2]

$$y'' + p(x)y = 0, \tag{4}$$

then

$$dc/dx_0 = y'^2(x_0)/y'^2(c), \tag{5}$$

where $y(x)$ *is any nonnull solution of* (4) *such that* $y(x_0) = y(c) = 0$.

Without loss in generality, we may assume that

$$y_1'(a) = y_2'(a) = 1,$$

and we first show that for δ positive and sufficiently small,

$$y_1(x) < y_2(x) \qquad (a < x \leqq a + \delta). \tag{6}$$

Note that

$$\lim_{x\to a} \frac{y_1''(x)}{y_2''(x)} = \lim_{x\to a} \frac{p_1(x)y_1(x)}{p_2(x)y_2(x)} = \frac{p_1(a)}{p_2(a)} > 1.$$

Since $y_i''(x) < 0$ $(i = 1, 2;\ a < x \leqq a + \delta)$, for δ small, we have $y_1''(x) < y_2''(x)$ $(a < x \leqq a + \delta)$. It follows that

$$\int_a^x y_1''(x)dx < \int_a^x y_2''(x)dx \qquad (a < x \leqq a + \delta);$$

accordingly,

$$y_1'(x) < y_2'(x) \qquad (a < x \leqq a + \delta). \tag{7}$$

A similar argument shows that (7) implies (6).

Consider next the "wronskian"

$$w(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$$

and its derivative

$$w' = (p_1 - p_2)y_1y_2.$$

Note that $w(a) = w(b) = 0$ and that near $x = a$, $w' > 0$, while near $x = b$, $w' < 0$. Further, $w'(x_0) = 0$ for x_0 on (a, b) if and only if $p_1(x_0) = p_2(x_0)$. Inasmuch as the curves $y = p_i(x)$ have a single intersection on (a, b) it follows that $w(x) > 0$ on (a, b). These observations lead to the conclusion that the curves $y = y_1(x)$ and $y = y_2(x)$ have no point in

[2] We assume $p(x)$ to be continuous on I.

common on (a, b), for, if there were such a point, there would be a first such point $x=x_1$. The fact that $w(x_1)=y_1(x_1)[y_2'(x_1)-y_1'(x_1)]$ would then be positive would imply that $y_2'(x_1)>y_1'(x_1)$—which is impossible because of (6). Thus,

$$y_1(x) < y_2(x) \qquad (a < x < b). \tag{8}$$

Next, we shall show that

$$|y_1'(b)| < |y_2'(b)|. \tag{9}$$

Note that

$$\frac{y_1''(x)}{y_2''(x)} = \frac{p_1(x)y_1(x)}{p_2(x)y_2(x)} < 1, \tag{10}$$

for $x<b$, near b; accordingly, $y_1''(x)>y_2''(x)$, near b, and

$$\int_x^b y_1''(x)dx > \int_x^b y_2''(x)dx.$$

It follows that

$$y_1'(b) - y_2'(b) > y_1'(x) - y_2'(x), \tag{11}$$

and an integration of (11) yields the fact that

$$y_1'(b) - y_2'(b) > \frac{y_2(x) - y_1(x)}{b - x},$$

for all $x<b$, sufficiently near b. Let x be any fixed number near b, and we have $y_1'(b)-y_2'(b)>0$; that is, (9) holds.

The proof of the theorem may now be completed by an appeal to the lemma. For, if c_1 and c_2 are conjugate points of $x=a$ with respect to (1) and (2), respectively, we have, when $c_i=b$,

$$dc_1/da = 1/y_1'^2(b),$$
$$dc_2/da = 1/y_2'^2(b).$$

Thus, at $x=b$,

$$dc_1/da > dc_2/da,$$

and the theorem is established.

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