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BOUNDED SOLUTIONS OF STIELTJES INTEGRAL EQUATIONS

DAVID LOWELL LOVELADY

ABSTRACT. Necessary and sufficient conditions are found for the existence of bounded solutions to some classes of nonhomogeneous linear Stieltjes integral equations. A theorem on the stability of bounded solutions is obtained, an application to a nonlinear Stieltjes integral equation is made.

I. **Introduction.** Let X be a Banach space with norm N_1. We wish to study the existence and stability of bounded solutions to

$$h(t) = f(t) + (L)\int_0^t dF[h],$$

where f is a function from $S=[0, \infty)$ to X, and F is a function from S to a space of Lipschitz operators on X. This equation has been studied by Neuberger [**10**] for F continuous and of bounded variation, and then by Mac Nerney [**6**] for F simply of bounded variation. Numerical bounds for its homogeneous solutions have been obtained by Martin [**8**] (see also [**7**]), and similar bounds for its nonhomogeneous solutions have been obtained by this author [**4**]. The study of bounded solutions for differential equations seems to have originated with Perron [**11**] and has been considerably extended by Bellman [**2**] and by Massera and Schäffer [**9**]. Most recently, Hallam [**3**] has studied relatively bounded solutions for differential equations, and Barbašin [**1**] has studied those Stieltjes integral equations which arise from differential equations with "delta-function" nonhomogeneities. We shall extend these results to the equation studied by Mac Nerney, in the linear case. Through the inequalities of [**8**] and [**4**], some of our results will be extended to the nonlinear case.

II. **Preliminaries.** Let H be the set to which A belongs only in case A is a function from X to X, $A[0]=0$, and there is a number b so that $N_1[A[p]-A[q]] \leqq bN_1[p-q]$ whenever each of p and q is in X. If A is in H, let $N_2[A]$ be the least number b so that $N_1[A[p]-A[q]] \leqq bN_1[p-q]$ whenever each of p and q is in X, and let $N_3[A]$ denote the least number b so that $N_1[A[p]] \leqq bN_1[p]$ whenever p is in X.

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Note that if A is in H and is linear, then $N_2[A]=N_3[A]$. Let BVH (respectively BVX) be the set to which F belongs only in case F is a function from S to H (respectively X) which is of bounded N_2-variation (respectively bounded N_1-variation) on each bounded interval of S. Let BVX_0 be that subset of BVX to which f belongs only in case $f(0)=0$. A member F of BVH will be called linear only in case each value of F is a linear operator. Let OM be the set to which W belongs only in case W is an H-valued function on that subset of $S\times S$ to which (t, s) belongs if and only if $t\geqq s$, and

(i) $W(x, y)W(y, z)=W(x, z)$ whenever $x\geqq y\geqq z\geqq 0$, where the multiplication is composition; and

(ii) whenever a is in S there is a number b so that if $a=t_0\geqq t_1\geqq t_2 \geqq \cdots \geqq t_n\geqq 0$, then

$$\sum_{k=1}^{n} N_2[W(t_{k-1}, t_k) - I] \leqq b,$$

where I in H is given by $I[p]=p$.

In [6], Mac Nerney showed that if F is in BVH, then there is exactly one member W of OM so that if a is in S, p is in X, and h is given by $h(t)=W(t, a)[p]$ if $t\geqq a$, $h(t)=0$ if $t<a$, then h is in BVX, and if $t\geqq a$,

$$h(t) = p + (L)\int_a^t dF[h],$$

where a one-term approximation of $(L)\int_a^t dF[h]$ is $F(t)[h(a)] - F(a)[h(a)]$. If F and W are related as in the previous sentence, W will be called "the fundamental solution for F."

The following lemma is very similar to Theorem 5.2 of [5], and we shall not prove it here.

LEMMA 2.1. *Let F be a linear member of BVH, and let W be the fundamental solution for F. Let p be in X, and let f be in BVX_0. Then, if h is in BVX, these are equivalent*:

(i) *$h(t)=p+f(t)+(L)\int_0^t dF[h]$ whenever t is in S.*

(ii) *$h(t)=W(t, 0)[p]+(R)\int_0^t W(t, \ )[df]$ if t is in S.*

III. **Bounded solutions for linear equations.** Let P be a nondecreasing function from S to S with $P(0)>0$. If t is in S, $P(t)^{-1}$ will denote the reciprocal of $P(t)$.

Let A be that subset of BVX_0 to which f belongs only in case

$$\sup_{t\geqq 0}\left[(R)\int_0^t P^{-1}N_1[df]\right]$$

is finite. If f is in A, let

$$J[f] = \sup_{t \geqq 0} \left[(R) \int_0^t P^{-1} N_1[df] \right].$$

Let C be that subset of BVX_0 to which f belongs only in case

$$\sup_{t \geqq 0} \left[P(t)^{-1} \limsup_{\delta \to 0} \left[(N_1[f(t+\delta) - f(t)])/\delta \right] \right]$$

is finite. If f is in C, let

$$L[f] = \sup_{t \geqq 0} \left[P(t)^{-1} \limsup_{\delta \to 0} \left[(N_1[f(t+\delta) - f(t)])/\delta \right] \right].$$

Now each of A and C is a linear space, each of J and L is a norm on its respective domain, and standard completeness arguments show that each of A and C is a Banach space with respect to its respective norm.

Suppose F is a linear member of BVH, and t is in S. Consider the linear function from BVX_0 to X which maps f to $(R)\int_0^t F[df]$. It follows from elementary inequalities that the restriction of this function to A is a continuous linear operator and has operator norm

$$\sup_{0 < s \leqq t} N_3[P(s)F(s)].$$

The restriction to C is also a continuous linear operator with operator norm $\int_0^t N_3[PF]d\rho$, where ρ from S to S is given by $\rho(s) = s$. This last observation is immediate if F is a left-continuous step function, and the general case follows easily. The integral $\int_0^t N_3[PF]d\rho$ need not be specified as either left or right (i.e. $(L)\int$ or $(R)\int$) since ρ is continuous.

In the sequel, R will be a function from S to S having only positive values, with each of R and R^{-1} having bounded variation on each bounded interval of S. A function f from S to X will be called R-bounded only in case there is a number b such that $N_1[f(t)] \leqq bR(t)$ whenever t is in S. R-boundedness for functions from S to S is defined analogously.

The following theorem is our main result and is a direct consequence of the Banach-Steinhaus Uniform Boundedness Theorem. The Banach-Steinhaus has been used in a similar fashion by Perron [**11**], Bellman [**2**], Massera and Schäffer [**9**], and Hallam [**3**]; all of these in the case of differential equations. For the Banach-Steinhaus Theorem itself, we refer to Rudin [**12**, p. 98].

THEOREM 3.1. *Let F be a linear member of BVH, and let W be the fundamental solution for F. If f is in BVX_0 and p is in X, let $T[f, p]$ be that member h of BVX such that if t is in S, then*

$$h(t) = p + f(t) + (L)\int_0^t dF[h].$$

Then (i) *and* (ii) *are equivalent,* (iii) *and* (iv) *are equivalent,* (v) *and* (vi) *are equivalent, and* (vii) *and* (viii) *are equivalent.*

(i) *$T[f, p]$ is R-bounded whenever f is in A and p is in X.*

(ii) *There is a number b such that if $t \geqq s > 0$, then*

$$N_3[P(s)W(t, s)] \leqq bR(t).$$

(iii) *There is a dense subset D of A such that if f is in D then $T[f, 0]$ is not R-bounded.*

(iv) *If b is a number, there are members t and s of S, $t \geqq s$, such that*

$$N_3[P(s)W(t, s)] > bR(t).$$

(v) *$T[f, p]$ is R-bounded whenever f is in C and p is in X.*

(vi) *There is a number b such that if t is in S, then*

$$\int_0^t N_3[PW(t, \ )]d\rho \leqq bR(t).$$

(vii) *There is a dense subset D of C such that if f is in D then $T[f, 0]$ is not R-bounded.*

(viii) *If b is a number, there is t in S such that*

$$\int_0^t N_3[PW(t, \ )]d\rho > bR(t).$$

REMARK. The first pair of equivalences includes parts of Theorems 5.3 and 5.5 of [**9**], and includes Theorem 1 of [**1**]. The second pair of equivalences includes parts of Theorems 5.2 and 5.3 of [**9**], and includes Theorem 1 of [**3**].

INDICATION OF PROOF. If t is in S, let Λ_t be the linear function from BVX_0 to X given by

$$\Lambda_t[f] = [R(t)]^{-1}\cdot(R)\int_0^t W(t, \ )[df] = R(t)^{-1}T[f, 0](t).$$

Now apply Rudin's formulation of the Banach-Steinhaus Theorem to the family $\{\Lambda_t : t \text{ is in } S\}$. It is clear that, after this application, to complete the proof we need only show that each of conditions (ii) and

(vi) imply $W(\ ,0)[p]$ is R-bounded whenever p is in X. This is immediate for condition (ii) since P is nondecreasing. Now suppose (vi) holds. Obviously $b \geqq 0$, and if $b=0$ we are through, so assume $b>0$. We take three cases.

Case 1. Suppose there is c in S such that $W(c, 0)=0$. Now $W(t, 0) = W(t, c)W(c, 0)=0$ whenever $t \geqq c$, so $N_3[W(\ ,0)]$ is R-bounded, and we are through.

Case 2. Suppose there is c in S and a sequence $(s_n)_{n=1}^{\infty}$ into S such that $s_n \to c$ and $N_3[W(s_n, 0)] \to 0$ as $n \to \infty$. We can, and do, assume $s_n \leqq c+1$ for each n. Let $t > c+1$. Find a number β such that $N_3[W(t, x)] \leqq \beta$ whenever $t \geqq x \geqq 0$. Now

$$\begin{aligned} N_3[W(t, 0)] &= \lim_{n \to \infty} N_3[W(t, s_n)W(s_n, 0)] \\ &\leqq \beta \cdot \lim_{n \to \infty} N_3[W(s_n, 0)] = 0, \end{aligned}$$

and thus $N_3[W(\ ,0)]$ is R-bounded.

Case 3. Suppose $N_3[W(\ ,0)]$ has only positive values and is bounded away from zero on each bounded interval of S. In this case the proof of [**3**, Lemma 1] can be easily modified so as to imply $N_3[W(\ ,0)]$ is R-bounded, so we shall not prove it here.

IV. **Stability.** Our stability result will be obvious from the results §III. The analogous result seems to have been largely overlooked in the case of differential equations.

THEOREM 4.1. *Let F, W, and T be as in Theorem* 3.1. *Then* (i) *and* (ii) *are true.*

(i) *If* (ii) *of Theorem* 3.1 *holds, then there are numbers d_1 and d_2 so that*

$$N_1[T[f, p](t) - T[g, q](t)] \leqq R(t)d_1N_1[p-q] + R(t)d_2J[f-g]$$

whenever f and g are in A, p and q are in X, and t is in S.

(ii) *If* (iv) *of Theorem* 3.1 *holds, then there are numbers d_1 and d_2 so that*

$$N_1[T[f, p](t) - T[g, q](t)] \leqq R(t)d_1N_1[p-q] + R(t)d_2L[f-g]$$

whenever f and g are in C, p and q are in X, and t is in S.

V. **The nonlinear equation.** Since X is a Banach space, the set of real numbers can be considered as a subset of H. Let BVR and OMR consist of the real-valued members of BVH and OM respectively. Martin [**8**, Lemma 3.3. Theorem 3.1, Theorem 3.3] (see also [**7**, Theorem 1, Theorem 2]) has proved the following lemma.

LEMMA 5.1. *Let F be in BVH. If $b \geqq a \geqq 0$, then $\int_a^b (N_3[I+dF]-1)$ exists. If γ is given by $\gamma(t) = \int_0^t (N_3[I+dF]-1)$, then γ is in BVR. If W and λ are fundamental solutions for F and γ respectively, then $N_3[W(t, s)] \leqq \lambda(t, s)$ whenever $t \geqq s$. Furthermore, λ is the least member of OMR for which this last inequality holds.*

In extending this to the nonhomogeneous equation, this author [4, Theorem A] proved the following lemma.

LEMMA 5.2. *Let F, W, γ, and λ be as in Lemma 5.1. Let f be in BVX_0, and let p be in X. Let h be that member of BVX such that if t is in S, then*

$$h(t) = p + f(t) + (L) \int_0^t dF[h].$$

Let G be that member of BVR such that if t is in S, then

$$G(t) = N_1[p] + \int_0^t N_1[df] + (L) \int_0^t (d\gamma)G.$$

Then $N_1[h(t)] \leqq G(t)$ whenever t is in S.

The following theorem is now an immediate consequence of Theorem 3.1 and Lemma 5.2.

THEOREM 5.1. *Let F, W, γ, and λ be as in Lemma 5.1. If f is in BVX_0 and p is in X, let $T[f, p]$ be that member h of BVX so that if t is in S, then*

$$h(t) = p + f(t) + (L) \int_0^t dF[h].$$

Then each of (i) *and* (ii) *is true.*

(i) *If there is a number b so that $P(s)\lambda(t, s) \leqq bR(t)$ whenever $t \geqq s > 0$, then $T[f, p]$ is R-bounded whenever f is in A and p is in X.*

(ii) *If there is a number b so that $\int_0^t P\lambda(t, \ )d\rho \leqq bR(t)$ whenever t is in S, then $T[f, p]$ is R-bounded whenever f is in C and p is in X.*

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Georgia Institute of Technology, Atlanta, Georgia 30332