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ON DIFFERENTIABILITY OF MINIMAL SURFACES AT A BOUNDARY POINT[1]

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ABSTRACT. Let $F(z)=\{u(z),\ v(z),\ w(z)\}$, $|z|<1$, represent a minimal surface spanning the curve $\Gamma: \{U(s),\ V(s),\ W(s)\}$, s being the arc length. Suppose Γ has a tangent at a point P. Then $F(z)$ is differentiable at this point if $U'(s)$, $V'(s)$, $W'(s)$ satisfy a Dini condition at P.

Let Γ be a closed rectifiable Jordan curve in Euclidean 3-space, and let $F(z)=\{u(z), v(z), w(z)\}$, defined in the disk $\{z:|z|\leqq 1\}$ $(z=x+iy=re^{i\theta})$, represent a generalized minimal surface spanning Γ, i.e.

(i) $u(z), v(z), w(z)$ are harmonic in $|z|<1$ and continuous in $|z|\leqq 1$;

(ii) x, y are isothermal parameters in $|z|\leqq 1$, i.e.

$$|F_x|^2 := u_x^2+v_x^2+w_x^2 = |F_y|^2 := u_y^2+v_y^2+w_y^2, \tag{1}$$

$$F_x\cdot F_y := u_xu_y+v_xv_y+w_xw_y = 0; \tag{2}$$

(iii) $F(e^{i\theta})$, $0\leqq\theta<2\pi$, is a homeomorphism of $|z|=1$ with Γ.

The components u, v, w of the vector F are the real parts of analytic functions in $|z|<1$:

$$\lambda(z) = u(z)+iu^*(z), \quad \mu(z) = v(z)+iv^*(z), \quad \nu(z) = w(z)+iw^*(z).$$

Recently various theorems dealing with the boundary behavior of conformal maps in the plane have been extended to minimal surfaces by J. C. C. Nitsche [2], D. Kinderlehrer [1], S. E. Warschawski [3], and other authors. Nitsche's paper contains a survey of prior work on the boundary behavior of minimal surfaces. The purpose of this note is to present a local result concerning differentiability of minimal surfaces at a given point on the boundary. In fact, our result extends a theorem of Warschawski on conformal mapping in the plane, namely Theorem 1 in [4].

THEOREM. *Suppose* $\{U(s),\ V(s),\ W(s)\}$ *denotes the parametric representation of* Γ *in terms of arc length. Assume* $P_0=\{U(s_0),\ V(s_0),\ W(s_0)\}$ *is a point of* Γ *and that* Γ *has a tangent at* P_0, *i.e.* $U'(s_0)$, $V'(s_0)$, $W'(s_0)$ *exist.*[2]

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[2] We assume $\{U'(s_0),\ V'(s_0),\ W'(s_0)\}$ represents the unit tangent to Γ at P_0.

Suppose that there exists a nondecreasing, continuous function $\omega(t) \geqq 0$, $0 \leqq t \leqq a$ $(a > 0)$, *such that*

$$\int_0^a \frac{\omega(t)}{t}\, dt < \infty$$

and

$$\begin{aligned} |U'(s) - U'(s_0)| &\leqq \omega(|s - s_0|), \\ |V'(s) - V'(s_0)| &\leqq \omega(|s - s_0|), \\ |W'(s) - W'(s_0)| &\leqq \omega(|s - s_0|), \end{aligned}$$

for all points $\{U(s), V(s), W(s)\}$ *in a neighborhood of* P_0 *at which* $U'(s)$, $V'(s)$, $W'(s)$ *exist.*[3]

Let $F(e^{i\theta_0}) = P_0$. *Then*

$$\lim_{z \to z_0} \frac{\lambda(z) - \lambda(z_0)}{z - z_0} = \lambda'(z_0) \qquad (z_0 = e^{i\theta_0})$$

exists for unrestricted approach in $|z| \leqq 1$ $(z \neq z_0)$, *and*

$$\lim_{z \to z_0} \lambda'(z) = \lambda'(z_0)$$

for z in any Stolz angle with vertex at z_0. *The same holds for* $\mu(z)$ *and* $\nu(z)$.[4]

PROOF. Without loss of generality we may assume $U'(s_0) = 1$, $V'(s_0) = 0$, $W'(s_0) = 0$. Under the conditions of the theorem Warschawski proved the following facts (see [3, Part II, §§2–7]):

There is an interval $[\theta_1, \theta_2]$ containing θ_0 in its interior, a constant $\alpha > 1$, and a sector $S = \{z = re^{i\theta} : 0 < r < 1, \theta_1 < \theta < \theta_2\}$ such that, if $\varphi(\zeta)$ maps $|\zeta| < 1$ conformally onto S $(\varphi(1) = e^{i\theta_0})$ and

$$\tilde{f} = \operatorname{Log}[(\lambda_\theta + \alpha) \circ \varphi] = \operatorname{Log}[\tilde{\lambda}_\theta + \alpha] \qquad \left(\lambda_\theta = \frac{\partial \lambda(re^{i\theta})}{\partial \theta}\right),$$

then $\lim_{\zeta \to 1} \operatorname{Im} \tilde{f}(\zeta)$ exists for unrestricted approach in $|\zeta| \leqq 1$ as well as $\lim_{\rho \uparrow 1} \tilde{f}(\rho) = \tilde{f}(1)$. The same holds for $i\tilde{g} = i[\mu_\theta/(\lambda_\theta + \alpha)] \circ \varphi = i \cdot [\tilde{\mu}_\theta/(\tilde{\lambda}_\theta + \alpha)]$ and $i\tilde{h} = i \cdot [\nu_\theta/(\lambda_\theta + \alpha)] \circ \varphi = i \cdot [\tilde{\nu}_\theta/(\tilde{\lambda}_\theta + \alpha)]$.[5] Let $\Phi(\zeta)$

[3] It should be noted that under the hypotheses of the theorem one can show the existence of a subarc γ containing P_0 in its interior and having the following property: $\Delta s \leqq c(P_1P_2)^-$ where c is a constant, $c > 1$, Δs is the length of the subarc of γ between $P_1, P_2 \in \gamma$, and (P_1P_2) is the chordal distance.

[4] The author wishes to express his indebtedness to the referee for his remark simplifying the statement of the theorem.

[5] Isothermal relations (1) and (2) are essential in obtaining these results.

$=\Phi_1(\zeta)+i\Phi_2(\zeta)$ ($\Phi_1=\text{Re }\Phi$, $\Phi_2=\text{Im }\Phi$) be holomorphic in $|\zeta|<1$. Assume

$$\lim_{\rho\uparrow 1}\Phi(\rho) = \Phi(1) = \Phi_1(1) + i\Phi_2(1) \tag{3}$$

exists, and

$$\lim_{\zeta\to 1}\Phi_2(\zeta) = \Phi_2(1) \tag{4}$$

for unrestricted approach in $|\zeta|<1$. Then by a theorem of Warschawski [5, p. 315, Theorem II] one has

$$\text{(i)}\qquad \lim_{\eta\to 0}\frac{1}{\eta}\int_0^\eta \{\exp[\Phi(e^{it}) - \Phi(1)]\}e^{it}\,dt = 1 \tag{5}$$

and

$$\text{(ii)}\qquad \lim_{\eta\to 0}\frac{1}{\eta}\int_0^\eta \{\exp[\Phi_1(e^{it}) - \Phi_1(1)] - 1\}\,dt = 0. \tag{6}$$

Also, it is readily seen from the proof of this theorem that

(iii) There exists a subarc $\tilde{\gamma}$ of $|\zeta|=1$ with midpoint $\zeta=1$ such that $\lim_{\rho\uparrow 1}\Phi(\rho e^{it})=\Phi(e^{it})$ exists for almost all $e^{it}\in\tilde{\gamma}$, $\Phi(e^{it})$ is integrable along $\tilde{\gamma}$, and

$$\text{(iv)}\qquad \lim_{\eta\to 0}\frac{1}{\eta}\int_0^\eta |\Phi(e^{it}) - \Phi(1)|^2\,dt = 0. \tag{7}$$

Since $\varphi'(\zeta)\neq 0$, we can define $\log\varphi'(\zeta)$ as a single valued analytic function in $|\zeta|<1$. By our remarks at the beginning, $\Phi_0(\zeta)=\bar{f}(\zeta)+\log\varphi'(\zeta)$ satisfies (3) and (4) and we can apply (5) to $\Phi_0(\zeta)$ to obtain

$$\lim_{\eta\to 0}\frac{1}{\eta}\int_0^\eta \{\exp[\text{Log}(\bar{\lambda}_\theta(e^{it}) + \alpha) + \log\varphi'(e^{it}) - \text{Log}(\bar{\lambda}_\theta(1) + \alpha) - \log\varphi'(1)]\}e^{it}\,dt = 1$$

which implies

$$\lim_{\eta\to 0}\frac{1}{\eta}\int_0^\eta \frac{(\bar{\lambda}_\theta(e^{it}) + \alpha)\varphi'(e^{it})}{(\bar{\lambda}_\theta(1) + \alpha)\varphi'(1)}e^{it}\,dt = 1.$$

Letting $\varphi(e^{i\eta})=e^{i\xi}$ and changing the variable of integration ($\varphi(e^{it})=e^{i\theta}$) we readily obtain

(8) $$\lim_{\xi\to\theta_0}\frac{1}{\xi-\theta_0}\int_{\theta_0}^{\xi}(\lambda_\theta(e^{i\theta})+\alpha)e^{i\theta}\,d\theta=(\lambda_\theta(e^{i\theta_0})+\alpha)e^{i\theta_0}.$$

Now,

(9) $$\begin{aligned} e^{i\theta_0}&\left[\frac{\lambda(e^{i\xi})-\lambda(e^{i\theta_0})}{\xi-\theta_0}\right]\\ &=\frac{1}{\xi-\theta_0}\int_{\theta_0}^{\xi}\lambda_\theta(e^{i\theta})(e^{i\theta_0}-e^{i\theta})\,d\theta+\frac{1}{\xi-\theta_0}\int_{\theta_0}^{\xi}\lambda_\theta(e^{i\theta})e^{i\theta}\,d\theta,\end{aligned}$$

since $\lambda(e^{i\theta})$ is absolutely continuous [**3**]. By (8), the second term in (9) approaches the limit $\lambda_\theta(e^{i\theta_0})e^{i\theta_0}$, and the first term approaches 0 as $\xi\to\theta_0$, since

$$\frac{1}{|\xi-\theta_0|}\int_{\theta_0}^{\xi}|\lambda_\theta(e^{i\theta})|\,|e^{i\theta_0}-e^{i\theta}|\,d\theta\leqq\frac{|e^{i\xi}-e^{i\theta_0}|}{\xi-\theta_0}\int_{\theta_0}^{\xi}|\lambda_\theta(e^{i\theta})|\,d\theta$$

and $\lambda_\theta(e^{i\theta})$ is integrable. Therefore,

(10) $$\lim_{\xi\to\theta_0}\frac{\lambda(e^{i\xi})-\lambda(e^{i\theta_0})}{\xi-\theta_0}=\lambda_\theta(e^{i\theta_0}).$$

From (10) it follows that

(11) $$\lim_{e^{i\theta}\to e^{i\theta_0}}\frac{\lambda(e^{i\theta})-\lambda(e^{i\theta_0})}{e^{i\theta}-e^{i\theta_0}}=\lambda'(e^{i\theta_0})$$

exists.

The function $(\lambda(z)-\lambda(e^{i\theta_0}))/(z-e^{i\theta_0})$ is holomorphic in $|z|<1$ and by (11) and by the fact that $\lambda(z)$ is continuous on $|z|=1$ it is bounded on $|z|=1$. The continuity of $\lambda(z)$ in $|z|\leqq 1$ also ensures that

$$\frac{\lambda(z)-\lambda(e^{i\theta_0})}{z-e^{i\theta_0}}=O\left(\frac{1}{|z-e^{i\theta_0}|}\right)\qquad\text{for } |z|<1.$$

Therefore, by a theorem of Phragmén-Lindelöf

$$((\lambda(z)-\lambda(e^{i\theta_0}))/(z-e^{i\theta_0})$$

is bounded in $|z|<1$. Hence, by a theorem of Lindelöf,

$$\lim_{z\to e^{i\theta_0}}\frac{\lambda(z)-\lambda(e^{i\theta_0})}{z-e^{i\theta_0}}=\lambda'(e^{i\theta_0})$$

for unrestricted approach in $|z|\leqq 1$. The second equation,

$\lim_{z\to z_0}\lambda'(z)=\lambda'(z_0)$ in any Stolz angle with vertex at z_0, is a well-known consequence of the first.

We can apply (7) to $i\tilde{g}(\zeta)$ and obtain

$$\lim_{\eta\to 0}\frac{1}{\eta}\int_0^{\eta}\left|\frac{\tilde{\mu}_\theta(e^{it})}{\tilde{\lambda}_\theta(e^{it})+\alpha}-\frac{\tilde{\mu}_\theta(1)}{\tilde{\lambda}_\theta(1)+\alpha}\right|^2 dt = 0. \tag{12}$$

Also, we can apply (6) to $2\tilde{f}(\zeta)$ and conclude that

$$\lim_{\eta\to 0}\frac{1}{\eta}\int_0^{\eta}\left\{\left|\frac{\tilde{\lambda}_\theta(e^{it})+\alpha}{\tilde{\lambda}_\theta(1)+\alpha}\right|^2-1\right\} dt = 0. \tag{13}$$

Thus,

$$\frac{1}{|\eta|}\int_0^{\eta}|\tilde{\lambda}_\theta(e^{it})+\alpha|^2\, dt \leqq M_0 \tag{14}$$

for $|\eta|\leqq\eta_0$ and some constant M_0.

By Schwarz's inequality,

$$\begin{aligned}
&\frac{1}{\eta}\int_0^{\eta}|\tilde{\mu}_\theta(e^{it})(\tilde{\lambda}_\theta(1)+\alpha)-\tilde{\mu}_\theta(1)(\tilde{\lambda}_\theta(e^{it})+\alpha)|\, dt\\
&\qquad\leqq\left(\frac{1}{\eta}\int_0^{\eta}\left|\frac{\tilde{\mu}_\theta(e^{it})}{\tilde{\lambda}_\theta(e^{it})+\alpha}-\frac{\tilde{\mu}_\theta(1)}{\tilde{\lambda}_\theta(1)+\alpha}\right|^2 dt\right)^{1/2}\\
&\qquad\cdot\left(|\tilde{\lambda}_\theta(1)+\alpha|^2\cdot\frac{1}{\eta}\int_0^{\eta}|\tilde{\lambda}_\theta(e^{it})+\alpha|^2\, dt\right)^{1/2};
\end{aligned} \tag{15}$$

(12), (14) and (15) imply

$$\lim_{\eta\to 0}\frac{1}{\eta}\int_0^{\eta}|\tilde{\mu}_\theta(e^{it})(\tilde{\lambda}_\theta(1)+\alpha)-\tilde{\mu}_\theta(1)(\tilde{\lambda}_\theta(e^{it})+\alpha)|\, dt = 0. \tag{16}$$

Since $\varphi'(e^{it})$ is bounded in a neighborhood of $\zeta=1$, we also have

$$\lim_{\eta\to 0}\frac{1}{\eta}\int_0^{\eta}|\tilde{\mu}_\theta(e^{it})(\tilde{\lambda}_\theta(1)+\alpha)-\tilde{\mu}_\theta(1)(\tilde{\lambda}_\theta(e^{it})+\alpha)|\,|\phi'(e^{it})|\, dt=0. \tag{17}$$

Changing the variable of integration, as in the case of $\lambda_\theta(e^{i\theta})$, one concludes from (17) that

$$\lim_{\xi\to\theta_0}\frac{1}{\xi-\theta_0}\int_{\theta_0}^{\xi}|\mu_\theta(e^{i\theta})(\lambda_\theta(e^{i\theta_0})+\alpha)-\mu_\theta(e^{i\theta_0})(\lambda_\theta(e^{i\theta})+\alpha)|\, d\theta = 0. \tag{18}$$

Thus,

$$(19)\quad \begin{aligned}&(\lambda_\theta(e^{i\theta_0}) + \alpha) \lim_{\xi\to\theta_0} \frac{1}{\xi - \theta_0} \int_{\theta_0}^{\xi} \mu_\theta(e^{i\theta})\, d\theta \\ &= \mu_\theta(e^{i\theta_0}) \lim_{\xi\to\theta_0} \frac{1}{\xi - \theta_0} \int_{\theta_0}^{\xi} (\lambda_\theta(e^{i\theta}) + \alpha)\, d\theta.\end{aligned}$$

By (10)

$$(20)\quad \lim_{\xi\to\theta_0} \frac{1}{\xi - \theta_0} \int_{\theta_0}^{\xi} (\lambda_\theta(e^{i\theta}) + \alpha)\, d\theta = \lambda_\theta(e^{i\theta_0}) + \alpha,$$

and $\lambda_\theta(e^{i\theta_0}) + \alpha \neq 0$.

Therefore (19) and (20) imply

$$(21)\quad \lim_{\xi\to\theta_0} \frac{1}{\xi - \theta_0} \int_{\theta_0}^{\xi} \mu_\theta(e^{i\theta})\, d\theta = \mu_\theta(e^{i\theta_0}).$$

From (21) one infers that

$$\lim_{z\to e^{i\theta_0}} \frac{\mu(z) - \mu(e^{i\theta_0})}{z - e^{i\theta_0}}$$

exists for unrestricted approach in $|z| \leqq 1$, exactly the same way as we showed this limit exists in the case of $\lambda(z)$. We deal with $\nu(z)$ in a similar fashion.

It should be noted that one may assume only the subarc γ to be rectifiable and obtain the same result with slight modification of our proof.

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