ON AN INTEGRAL FORMULA OF
GAUSS-BONNET-GROTEMeyer

BANG-YEN CHEN

Abstract. Let $e(p)$ and $G(p)$ be the unit outer normal and the
Gauss-Kronecker curvature of an oriented closed even-dimensional
hypersurface $M$ of dimension $n$ in $E^{n+1}$. Then for a fixed
unit vector $c$ in $E^{n+1}$, we have

\[ \int_M (c\cdot e)^mGdV = c_{n+m}\chi(M)/c_m, \quad \text{for } m = 0, 2, 4, \cdots, \]
\[ = 0, \quad \text{for } m = 1, 3, 5, \cdots, \]

where $c\cdot e$ denotes the inner product of $c$ and $e$, $c_m$ the area of $m$-
dimensional unit sphere, and $\chi(M)$ the Euler characteristic of $M$.

Let $M$ be an orientable closed hypersurface imbedded in a eu-
clidean space $E^{n+1}$ of dimension $n+1 \geq 3$. Let $x(p)$ be the position
vector of a point $p$ with respect to a fixed point 0 in $E^{n+1}$, and $e(p)$,
$G(p)$ and $dV$ the unit outer normal, the Gauss-Kronecker curvature
at $p$, and the volume element of $M$ in $E^{n+1}$, respectively. The main
results of this paper are the following:

Theorem 1. Let $M$ be an oriented closed hypersurface of dimension $n$
imbedded in euclidean space $E^{n+1}$ of dimension $n+1 \geq 3$. Then we have

\[ m \int_M (x\cdot e)^{m-1}xGdV = (n + m) \int_M (x\cdot e)^m eGdV, \]
\[ m = 0, 1, 2, 3, \cdots. \]

Theorem 2. Under the same hypothesis of Theorem 1, if the dimension
of $M$ is even, then for a fixed unit vector $c$ in $E^{n+1}$, we have

\[ \int_M (c\cdot e)^mGdV = c_{n+m}\chi(M)/c_m, \quad \text{for } m = 0, 2, 4, \cdots, \]
\[ = 0, \quad \text{for } m = 1, 3, 5, \cdots. \]

Remark. If $m = 0$, then formula (3) is the well-known Gauss-
Bonnet formula, and if $m = 2$ and $n = 2$, then formula (3) was proved
1. Preliminaries. Let $M$ be an oriented (differentiable) manifold of dimension $n$, and let $x : M \to \mathbb{E}^{n+1}$ be a hypersurface. Let $e(p)$, $p \in M$, be the unit outer normal at $x(p)$. We consider the orthonormal frames $e_1, \ldots, e_n$ in the tangent hyperplane at $x(p)$, such that the determinant $(e_1, \ldots, e_n, e) = +1$. The space of all $e_1, \ldots, e_n$ can be identified with the principal fibre bundle $B$ of $M$ relative to the induced metric $dx \cdot dx$ (for the details, see Chern [2]). We have

\[ dx = \omega_1 e_1 + \cdots + \omega_n e_n, \quad de = \theta_1 e_1 + \cdots + \theta_n e_n, \]

so that $\omega_i, \theta_i, 1 \leq i \leq n$, are linear differential forms in $B$. Since

\[ e \cdot dx = 0, \]

we get, by exterior differentiation,

\[ de \wedge dx = 0. \]

The left-hand side in (6) is the exterior product of two vector-valued linear differential forms; vectors are multiplied in the sense of scalar products in $\mathbb{E}^{n+1}$. In view of (4), equation (6) can be written

\[ \sum_i \omega_i \wedge \theta_i = 0. \]

Since $\omega_i$ are linear independent, we can put, in view of (7),

\[ \theta_i = \sum A_{ij} \omega_j, \quad A_{ij} = A_{ji}, \quad 1 \leq i, j \leq n. \]

The Gauss-Kronecker curvature $G$ is given by

\[ G = \det(A_{ij}). \]

Since $e_1, \ldots, e_n$ is an orthonormal frame, we know that the volume element $dV = \omega_1 \wedge \cdots \wedge \omega_n$. Hence, by (8) and (9), we have

\[ \theta_1 \wedge \cdots \wedge \theta_n = G dV. \]

For simplicity, let $[\ldots, \ldots]$ ($n$ terms) denote the combining operation of the vector product of $\mathbb{E}^{n+1}$ with the exterior product. From (10), we have

\[ (n \text{ times}) \]

\[ [de, \ldots, de] = (n! GdV)e. \]

2. Proof of Theorem 1. Put

\[ \delta = \sum_i (-1)^{i-1} \theta_1 \wedge \cdots \wedge \theta_i \wedge \cdots \wedge \theta_n e_i, \]

where "\text{\textasciicircum}" denotes the omitted term. Then, from (4), we have
(n - 1 times)\[de, \ldots, de, e\] = [\sum \theta_i e_i, \ldots, \sum \theta_i e_i, e]

(13) = (n - 1)! \sum \theta_1 \wedge \ldots \wedge \theta_i \wedge \ldots \wedge \theta_n [e_1, \ldots, e_i, \ldots, e_n, e]

= (n - 1)! \sum (-1)^{n-i-1} \theta_1 \wedge \ldots \wedge \theta_i \wedge \ldots \wedge \theta_n e_i

= (n - 1)!(-1)^n \delta.

From (11) and (12), we get

(n times)

(14) \[d\delta = - [de, \ldots, de]/(n - 1)!] = -(nGdV)e.

By (4), (12) and (14), we have

\[d((x \cdot e)^m \delta) = m(x \cdot e)^{m-1}(x \cdot de) \wedge \delta + (x \cdot e)^m d\delta\]

(15) = m(x \cdot e)^{m-1} \sum (x \cdot e) \theta_1 e_1 \wedge \ldots \wedge \theta_n e_n + (x \cdot e)^m d\delta

= m(x \cdot e)^{m-1} xGdV - (n + m)(x \cdot e)^m eGdV.

Integrating both sides of (15) over \(M\) and applying Stokes' theorem, we get (2). This completes the proof of the theorem.

3. Proof of Theorem 2. Let \(c\) be a unit vector in \(E^{n+1}\). Taking the scalar product of \(c\) with both sides of (2), we get

(A0) \[m \int_M (x \cdot e)^{m-1}(x \cdot c)GdV = (n + m) \int_M (x \cdot e)^m(c \cdot e)GdV.\]

We make the translation \(x \rightarrow x + c\) of \(M\). Then, by (A0), we get

\[m \int_M \sum_{i=1}^{m-1} \binom{m-1}{i}(x \cdot e)^i(c \cdot e)^{m-i-1}((x \cdot c) + 1)GdV\]

(A0') = (n + m) \int_M \sum_{i=0}^{m} \binom{m}{i}(x \cdot e)^i(c \cdot e)^{m-i+1}GdV.

(A0') - (A0) gives

\[m \int_M \sum_{i=0}^{m-2} \binom{m-1}{i}(x \cdot e)^i(x \cdot c)(c \cdot e)^{m-i-1}GdV\]

(A1) + \(m \int_M \sum_{i=1}^{m-1} \binom{m-1}{i}(x \cdot e)^i(c \cdot e)^{m-i-1}GdV\]

= (n + m) \int_M \sum_{i=0}^{m} \binom{m}{i}(x \cdot e)^i(c \cdot e)^{m-i+1}GdV.

Again we make the translation \(x \rightarrow x + c\) of \(M\) into (A1) and then subtract from (A1), we get
Continuing this process $k$ times ($k = 1, 2, \ldots, m$), we get

$$m \int_M \sum_{i_1=0}^{m-2} \binom{m-1}{i_1} \sum_{i_2=0}^{i_1-1} \binom{i_1}{i_2} (x \cdot e)^{i_2}(x \cdot c)(c \cdot e)^{m-i_2-1}GdV$$

$$+ m \int_M \left[ \sum_{i_1=0}^{m-2} \binom{m-1}{i_1} \sum_{i_2=0}^{i_1} \binom{i_1}{i_2} + \sum_{i_1=0}^{m-1} \binom{m-1}{i_1} \sum_{i_2=0}^{i_1} \binom{i_1}{i_2} \right]$$

$$(A_k)$$

$$\cdot (x \cdot e)^{i_2}(x \cdot e)(c \cdot e)^{m-i_2-1}GdV$$

$$= (n + m) \int_M \sum_{i_1=0}^{m-1} \binom{m-1}{i_1} \sum_{i_2=0}^{i_1} \binom{i_1}{i_2} (x \cdot e)^{i_2}(x \cdot e)(c \cdot e)^{m-i_2+1}GdV.$$
Hence, we get

\[(17) \quad \int_M (c \cdot e)^m GdV = \frac{m - 1}{n + m - 1} \int_M (c \cdot e)^{m-2} GdV.\]

By the assumption, \(n\) is even. Hence if \(m\) is a positive even integer, then by (17), the Gauss-Bonnet formula and the fact

\[(18) \quad c_N = 2[\Gamma(\frac{1}{2})]^{N+1}/\Gamma(\frac{1}{2}(N + 1)),\]

we get

\[(19) \quad \int_M (c \cdot e)^m GdV = \frac{(m - 1)(m - 3) \cdots 1}{(n + m - 1)(n + m - 3) \cdots (n + 1)} \int_M GdV = c_{n+m}(M)/c_m.\]

Moreover, by (2), we get

\[(20) \quad \int_M e GdV = 0.\]

Taking the inner product of \(c\) with (20), we get

\[(21) \quad \int_M (c \cdot e) GdV = 0.\]

Hence, in view of (16) and (21), we find that

\[(22) \quad \int_M (c \cdot e)^m GdV = 0, \quad \text{for all } m = 1, 3, 5, \cdots.\]

Therefore, by (19), (22) and the Gauss-Bonnet formula, we get formula (3). This completes the proof of the theorem.

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REFERENCES


UNIVERSITY OF NOTRE DAME, NOTRE DAME, INDIANA 46556

MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48823