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PRODUCTS OF $\mathfrak{M}$-COMPACT SPACES[1]

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ABSTRACT. Some results are given on the closure under suitably restricted products of a class of spaces similar to one considered by Z. Frolík and, more recently, by N. Noble. An answer is given to the following question of Gulden, Fleischman, and Weston: Does there exist $\mathfrak{M}>\aleph_0$ and an $\mathfrak{M}$-compact space X such that some subset A of X of cardinality $\leqq\mathfrak{M}$ is contained in no compact subset of X? It is shown that for every $\mathfrak{M}\geqq\aleph_0$ there is a topological group which has this property.

1. **Preliminaries.** Throughout this paper all hypothesized cardinals and all hypothesized spaces will be infinite and T_1, respectively.

A filter base will be called *open* if the sets belonging to it are open, and the adherence of a filter base $\mathfrak{F}$ on a space X, $\bigcap\{\overline{F} \mid F\in\mathfrak{F}\}$, will be denoted by $\mathrm{ad}_X\,\mathfrak{F}$ or ad $\mathfrak{F}$. The cardinality of a set A will be denoted by $|A|$. [CH] indicates the Continuum Hypothesis is being assumed.

We are grateful to W. W. Comfort for several useful suggestions concerning this paper. The terminology "weakly-$\mathfrak{M}$-$\aleph_0$-compact" is due to the referee.

2. **Products of $\mathfrak{M}$-compact spaces.** In [9] Noble studies properties of $\mathfrak{C}^*$, the family of spaces in which every infinite subset meets some compact subset in an infinite set.

We shall call a space X *strongly $\mathfrak{M}$-compact* provided that it has the property: for every filter base $\mathfrak{F}$ on X with $|\mathfrak{F}|\leqq\mathfrak{M}$, there is a compact subset K of X such that $\mathfrak{F}|K$ is a filter base. A space is strongly $\aleph_0$-compact if and only if it belongs to $\mathfrak{C}^*$. (By a *strongly countably compact* space certain other authors, including J. Keesling and N. Noble, mean instead a space which has the property that below is called *$\aleph_0$-bounded.*)

As usual, a space X will be called *$\mathfrak{M}$-compact* provided that it has

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one of the equivalent properties: if $\mathfrak{U}$ is an open cover of X and $|\mathfrak{U}| \leqq \mathfrak{M}$, then $\mathfrak{U}$ has a finite subcover; for every filter base $\mathfrak{F}$ on X such that $|\mathfrak{F}| \leqq \mathfrak{M}$, ad $\mathfrak{F} \neq \emptyset$.

In [6] a space X is said to be $\mathfrak{M}$-*bounded* if for every subset A of X with $|A| \leqq \mathfrak{M}$, there is a compact subset K of X such that $A \subset K$.

The authors of [6] observe that an $\mathfrak{M}$-bounded space is $\mathfrak{M}$-compact. Other easily checked facts are the following: an $\mathfrak{M}$-bounded space is strongly $\mathfrak{M}$-compact; a strongly $\mathfrak{M}$-compact space is $\mathfrak{M}$-compact; and the property $\mathfrak{M}$-bounded (strongly $\mathfrak{M}$-compact, $\mathfrak{M}$-compact) is closed hereditary. They note that an example of Novák [10] is countably compact but not $\aleph_0$-bounded, and they ask if there is an $\mathfrak{M}$-compact space, $\mathfrak{M} > \aleph_0$, which is not $\mathfrak{M}$-bounded. In §4 it will be shown that for every (regular) $\mathfrak{M} \geqq \aleph_0$ there is an $\mathfrak{M}$-compact topological group which is not (strongly $\mathfrak{M}$-compact) $\mathfrak{M}$-bounded. We can show, though, that the concepts $\mathfrak{M}$-bounded and $\mathfrak{M}$-compact are more closely related than one might suspect.

THEOREM 2.1. *Let X be a regular space which is $\mathfrak{M}$-compact, and suppose that $\mathfrak{N}$ is any cardinal for which $2^{\mathfrak{N}} \leqq \mathfrak{M}$. Then X is $\mathfrak{N}$-bounded.*

PROOF. It is known (and easy to prove) that a regular space is compact if every open filter base has an adherent point.

Suppose that $A \subset X$ with $|A| \leqq \mathfrak{N}$, and let $\mathfrak{F}$ be any open filter base on $\bar{A}$. The filter base $\mathfrak{G} = \mathfrak{F}|A$ satisfies $|\mathfrak{G}| \leqq 2^{\mathfrak{N}} \leqq \mathfrak{M}$, so $\emptyset \neq \mathrm{ad}_{\bar{A}}\, \mathfrak{G} = \mathrm{ad}_{\bar{A}}\, \mathfrak{F}$.

REMARK 2.2. The following examples show that, for $\mathfrak{M} = \aleph_0$, the concepts $\mathfrak{M}$-bounded, strongly $\mathfrak{M}$-compact, and $\mathfrak{M}$-compact cover distinct classes. The first two are strongly $\mathfrak{M}$-compact but not $\mathfrak{M}$-bounded. The third is $\mathfrak{M}$-compact but not strongly $\mathfrak{M}$-compact.

(i) Let $K_x = \beta N - \{x\}$, where $x \in \beta N - N$. Frolík [3] notes that $K_x \in \mathfrak{C}^*$.

(ii) Mary Ellen Rudin [11] has shown that there exists [CH] a separable, noncompact, sequentially compact space.

(iii) Frolík [2] (and others) have constructed infinite countably compact spaces in which every compact subset is finite.

We now obtain two product theorems. The first is, for $\mathfrak{M} = \aleph_0$, due to Noble [9]. The second is a strengthening of the following theorem of A. H. Stone [12]: The product of not more than $\aleph_1$ sequentially compact spaces is countably compact.

THEOREM 2.3. *The product of an $\mathfrak{M}$-compact space and a strongly $\mathfrak{M}$-compact space is $\mathfrak{M}$-compact.*

The proof is immediate if one recalls that the product of a compact space and an $\mathfrak{M}$-compact space is $\mathfrak{M}$-compact ([2], [5]).

While $\mathfrak{M}$-boundedness (strong $\mathfrak{M}$-compactness) is obviously productive (finitely productive), the authors do not know, for $\mathfrak{M} > \aleph_0$, if either of the properties $\mathfrak{M}$-compact and strongly $\mathfrak{M}$-compact is productive.

In [2] and [3] Frolík shows (i) there exists $\mathfrak{F} \subset \mathfrak{C}^*$ such that $|\mathfrak{F}| = \mathfrak{c}$ but $\prod \mathfrak{F} \notin \mathfrak{C}^*$, and (ii) there exists $\mathcal{G} \subset \mathfrak{C}^*$ such that $|\mathcal{G}| = 2^{\mathfrak{c}}$ and $\prod \mathcal{G}$ is not countably compact ($\mathcal{G} = \{K_x \mid x \in \beta N - N\}$). His proof of 4.2.3 in [3] shows that $\mathfrak{C}^*$ is closed under countable products.

THEOREM 2.4. *The product of not more than $\aleph_1$ spaces in $\mathfrak{C}^*$ is countably compact.*

PROOF. Let $X = \prod X_a$ where a runs over the set of all countable ordinals, and where each $X_a \in \mathfrak{C}^*$. Given a sequence $\{x(n) \mid n \in N\}$ of points of X, we produce a cluster point. Let S be the set of all infinite subsets of N.

A proof about the same as A. H. Stone's proof of [12, Theorem 5.5] shows that there exist sets $s(a) \in S$ for each a, such that

(1) $K_a = \{x_a(n) \mid n \in s(a)\}^-$ is compact;

(2) whenever $b < a$, $s(a) - s(b)$ is finite.

Let K be the compact set $\prod K_a$, and suppose that no point of K is a cluster point of $\{x(n)\}$. Then there are a finite number of basic open sets $U(i)$ such that $K \subset U = \bigcup \{U(i)\}$ and for some j, $U \cap \{x(n) \mid n \in N \text{ and } n \geqq j\} = \emptyset$. But consider the set F of all b for which some $\mathrm{pr}_b(U(i)) \neq X_b$. F is finite, say, $F = \{b_1, \cdots, b_k\}$ where each $b_i < b_{i+1}$. By (2) there exists $p \in s(b_k)$ such that for every $n \in s(b_k)$, $n \geqq p$ implies that $n \in s(b_i)$, $i = 1, \cdots, k-1$. By (1) there is, for any such n, a point $t(n) \in K$ which satisfies $t_b(n) = x_b(n)$ for every $b \in F$. Since each $t(n)$ belongs to some $U(i)$, and since, clearly, $t(n) \in U(i)$ implies that $x(n) \in U(i)$, it follows that U contains $\{x(n) \mid n \in s(b_k)$ and $n \geqq p\}$. This is a contradiction.

3. **Products of weakly-$\mathfrak{M}$-$\aleph_0$-compact spaces.** We shall call a space X *weakly-$\mathfrak{M}$-$\aleph_0$-compact* provided that one of the following equivalent conditions holds: if $\mathfrak{U}$ is an open cover of X such that $|\mathfrak{U}| \leqq \mathfrak{M}$, then there is a finite subfamily $\mathfrak{V}$ of $\mathfrak{U}$ such that $X = [\bigcup \mathfrak{V}]^-$; if $\mathfrak{F}$ is an open filter base on X such that $|\mathfrak{F}| \leqq \mathfrak{M}$, then ad $\mathfrak{F} \neq \emptyset$.

Weakly-$\aleph_0$-$\aleph_0$-compactness is the same as A. H. Stone's *feeble compactness* [12] (and, for completely regular spaces, is the same as pseudocompactness). At the other end of the spectrum are the *weakly-$\aleph_0$-compact spaces*—the spaces on which all open filter bases have nonempty adherence. In [1] Frolík introduced weakly-$\mathfrak{M}$-$\aleph_0$-compact spaces (under a different name) and obtained several interesting analogues of known theorems about weakly-$\aleph_0$-compact spaces.

Although the property weakly-$\aleph_0$-compact is productive [7], it is well known that there exist two pseudocompact (in fact, countably compact) completely regular spaces whose product is not pseudocompact.

The results of this section show that if the Generalized Continuum Hypothesis holds, then for every regular cardinal $\mathfrak{M}$, there exist weakly-$\mathfrak{M}$-$\aleph_0$-compact completely regular spaces M_1 and M_2 such that $M_1 \times M_2$ is not weakly-$\mathfrak{M}$-$\aleph_0$-compact. Our construction will be similar to ones due to Frolík [2], Novák [10], and Terasaka [4, 9.15].

For the remainder of this section, $\mathfrak{M}$ will be an arbitrary but fixed regular cardinal which has the following property: for every $\mathfrak{N} < \mathfrak{M}$, $2^{\mathfrak{N}} \leqq \mathfrak{M}$. X will be a discrete space with $|X| = \mathfrak{M}$. For $V \subset X$, we shall write V^* for the clopen set $\mathrm{Cl}_{\beta X} V$. Y will denote $\{x \in \beta X \mid \text{for some } V \subset X, |V| < \mathfrak{M} \text{ and } x \in V^*\}$, and Z will denote $\{x \in \beta X \mid \text{for every } V \subset X, x \in V^* \text{ implies } |V| = \mathfrak{M}\}$.

A subset D of βX will be called *strongly discrete* if there exist sets $V_d \subset X$, $d \in D$, so that $d \in V_d^*$, and for all $d, e \in D$, $d \neq e$ implies that $V_d \cap V_e = \varnothing$.

Lemma 3.1. *Let D be a strongly discrete set such that $|D| = \mathfrak{M}$. Denote by D' the set of all points $z \in Z \cap \overline{D}$ such that for every subset E of D, if $|E| < \mathfrak{M}$, then $z \notin \overline{E}$. Then $|D'| = 2^{2^{\mathfrak{M}}}$.*

Proof. An argument like the one on p. 91 of [4] shows that $\overline{D} = \beta D$. Thus $|\overline{D}| = 2^{2^{\mathfrak{M}}}$ [4, p. 130].

Well-order $D \cup X$ as $\{x_a \mid a \in \mathfrak{M}\}$ and for each $b \in \mathfrak{M}$ let $S(b) = \{x_a \mid a \leqq b\}$. Since $\mathfrak{M}$ is regular, $\overline{D} - D' \subset \bigcup \{S(b)^- \mid b \in \mathfrak{M}\}$. Because $\mathfrak{N} < \mathfrak{M} \Rightarrow 2^{\mathfrak{N}} \leqq \mathfrak{M}$, each $|S(b)^-| \leqq 2^{\mathfrak{M}}$. Thus $|\overline{D} - D'| \leqq 2^{\mathfrak{M}}$.

Remark 3.2. The following example shows that in Lemma 3.1, for $\mathfrak{M} = \mathfrak{c}$, strongly discrete cannot be replaced by discrete. It is known (see [8] or [13]) that [CH] there is a one-to-one mapping f of ω_1 into $\beta N - N$ and a point $p \in \beta N - N - f(\omega_1)$ such that the net f converges to p and $f(\omega_1)$ is a discrete subset of βN. Thus each nonisolated point of $[f(\omega_1)]^- - \{p\}$ is a limit point of a countable subset of $f(\omega_1)$. It is also known that for any infinite discrete space X, $\beta X - X$ contains a copy of βN.

In [14] Grant Woods has independently obtained a result which, for the case $\mathfrak{M} = \mathfrak{N}^+$, generalizes our next lemma.

Lemma 3.3. *The space Y is $\mathfrak{N}$-bounded for every $\mathfrak{N} < \mathfrak{M}$.*

Proof. Let $C \subset Y$ with $|C| \leqq \mathfrak{N}$. For each $c \in C$ there is a set $V_c \subset X$ with $c \in V_c^*$ and $|V_c| < \mathfrak{M}$. Then $V = \bigcup \{V_c\} \subset X$ and $|V| < \mathfrak{M}$, so $C \subset V^* \subset Y$.

LEMMA 3.4. *There exist disjoint subsets P_1 and P_2 of Z so that whenever D and D' are as in the hypothesis of Lemma* 3.1, *then* $D'\cap P_1 \neq \emptyset \neq D'\cap P_2$.

PROOF. Let $\mathfrak{F} = \{D \mid |D| = \mathfrak{M}$ and D is strongly discrete$\}$. Then $|\mathfrak{F}| \leqq 2^{2^{\mathfrak{M}}}$. Since for each $D \in \mathfrak{F}$, $|D'| = 2^{2^{\mathfrak{M}}}$, one can construct P_1, P_2 by induction.

THEOREM 3.5. *Let* $M_i = Y \cup P_i$, $i = 1, 2$.

(i) *Each M_i is weakly-$\mathfrak{M}$-$\aleph_0$-compact.*

(ii) *$M_1 \times M_2$ is not weakly-$\mathfrak{M}$-$\aleph_0$-compact.*

PROOF. (i). It suffices to show that an open filter base $\mathfrak{F}$ on X with $|\mathfrak{F}| \leqq \mathfrak{M}$ has an adherent point.

If $|\mathfrak{F}| < \mathfrak{M}$, select $x_F \in F$ for each $F \in \mathfrak{F}$. Then $\{x_F\}^-$ is a compact subset of Y by Lemma 3.3, so $\mathfrak{F}$ has an adherent point $y \in Y \subset M_1 \cap M_2$.

Suppose that $\mathfrak{M} = |\mathfrak{F}|$ and well-order $\mathfrak{F}$ as $\{F_a \mid a \in \mathfrak{M}\}$. As above, we can, for each $d \in \mathfrak{M}$, select $y_d \in Y$ so that $y_d \in \bigcap\{[F_a]^- \mid a \leqq d\}$.

Case 1. There is a set $V \subset X$ with $|V| < \mathfrak{M}$ so that the net $\{y_a \mid a \in \mathfrak{M}\}$ is frequently in V^*. Then $V^* \subset Y$ and there must be a point $y \in V^*$ which is a cluster point of $\{y_a\}$. The point y would be an adherent point of $\mathfrak{F}$.

Case 2. We suppose that for every set $V \subset X$ such that $|V| < \mathfrak{M}$, the net $\{y_a\}$ is eventually in $Y - V^*$.

Then an inductive argument shows that there exist a nondecreasing mapping f of $\mathfrak{M}$ into itself and sets $V_{f(a)} \subset X$ for each a, such that

(1) $|V_{f(a)}| < \mathfrak{M}$;

(2) $y_{f(a)} \in V^*_{f(a)}$;

(3) $a \leqq f(a)$;

(4) whenever $a \neq b$, $V_{f(a)} \cap V_{f(b)} = \emptyset$.

Let $D = \{y_{f(a)} \mid a \in \mathfrak{M}\}$. Then $|D| = \mathfrak{M}$ and D is strongly discrete, so $D' \cap P_i \neq \emptyset$, $i = 1$ and $i = 2$. Any point in $D' \cap P_i$ is an adherent point of $\mathfrak{F}$ in M_i.

(ii). Well-order X as $\{x_a \mid a \in \mathfrak{M}\}$, and for each $b \in \mathfrak{M}$ let $F_b = \{(x_a, x_a) \mid b \leqq a\}$. Then $\mathfrak{F} = \{F_b \mid b \in \mathfrak{M}\}$ is an open filter base on $M_1 \times M_2$ and $|\mathfrak{F}| = \mathfrak{M}$, but $\emptyset = \text{ad } \mathfrak{F}$, because any adherent point of $\mathfrak{F}$ would have to be of the form (z, z) for some $z \in Z$.

4. $\mathfrak{M}$-compact spaces and groups. In this section we consider various ways to construct groups and completely regular spaces that are (i) noncompact and $\mathfrak{M}$-bounded and (ii) $\mathfrak{M}$-compact but not strongly $\mathfrak{M}$-compact.

4.1. In [6] a point p of a space Y is called an $\mathfrak{M}$-*point* of Y provided that for every family $\mathfrak{F}$ of open subsets of Y, if $|\mathfrak{F}| \leqq \mathfrak{M}$ and $p \in \bigcap \mathfrak{F}$, then $\bigcap\mathfrak{F}$ is a neighborhood of p. According to Theorem 2 of [6], if Y is a subspace of a compact space X and if each point of $X\backslash Y$ is an $\mathfrak{M}$-point of X, then Y is $\mathfrak{M}$-bounded. The following are two ways in which one can use this result to obtain spaces of the type (i).

(a) Given a nonlimit ordinal $a>0$, the space ω_a with the usual topology is an $\aleph_{a-1}$-bounded but not weakly-$\aleph_a$-$\aleph_0$-compact space in which each point has a fundamental system of neighborhoods of cardinality $\leqq\aleph_{a-1}$.

(b) Let G be a nonempty set of P-points (i.e., $\aleph_0$-points) [CH] of $\beta\boldsymbol{N}-\boldsymbol{N}$ and take $Y=\beta\boldsymbol{N}-\boldsymbol{N}-G$. Since the non-$P$-points form a dense subset of $\beta\boldsymbol{N}-\boldsymbol{N}$, Y is $\aleph_0$-bounded but not weakly-$\mathfrak{c}$-$\aleph_0$-almost compact.

The next result shows that the method in (b) cannot be used for nonmeasurable cardinals $\geqq\mathfrak{c}$.

THEOREM 4.2. *Let X be a discrete space with $|X|=\mathfrak{M}$. If $\mathfrak{M}$ is nonmeasurable and $\geqq\mathfrak{c}$, then no point of $\beta X-X$ is an $\mathfrak{M}$-point of $\beta X-X$.*

PROOF. Let us first observe that (1) no point of $\beta X-X$ is a P-point of the space βX, and (2) if p is an $\mathfrak{M}$-point of $\beta X-X$, then for every $V\subset X, p\in V^*\Rightarrow|V|>\aleph_0$.

(1) Since X is realcompact, for each $p\in\beta X-X$ there is a continuous real valued function f defined on βX such that $f(p)=0$ and $f(X)>0$. On the other hand, if p is a P-point of a space T, then for every continuous real valued function f defined on T, there is a neighborhood of p in T on which f is constant.

(2) If $|V|=\aleph_0$ and $p\in V^*$, then $\{p\}$ is an intersection of $\mathfrak{c}$ clopen subsets of $\beta X-X$.

Next, suppose that there exists an $\mathfrak{M}$-point of $\beta X-X$, say, p. Using (2), we will show that p is a P-point of βX, thereby contradicting (1).

Let $\{V_n^*|n\in\boldsymbol{N}\}$ be sets containing p. By hypothesis there is a subset V of X so that $p\in V^*$ and $V^*-X\subset\bigcap\{(V_n^*-X)|n\in\boldsymbol{N}\}$. Now every set $V-V_n$ must be finite since each $V_n^*-X\supset V^*-X$. Let W denote $\bigcup\{V-V_n|n\in\boldsymbol{N}\}$. Then W is countable and $p\in V^*=(V-W)^*\cup W^*$. By (2), $p\in(V-W)^*$. Since $V-W\subset V_n$ for every n, $(V-W)^*\subset\bigcap\{V_n^*\}$.

We do not know if there exists $p\in\beta X-X$ such that for every subset A of $\beta X-X-\{p\}$, $|A|\leqq\mathfrak{M}\Rightarrow p\notin\overline{A}$.

4.3. Let $\mathfrak{N}$ be a cardinal and $\mathfrak{M}$ its successor. The following space, C, similar to ones due to H. H. Corson, I. Glicksberg, L. S. Pontryagin and J. Kister, is $\mathfrak{N}$-bounded but is not, in general, $\mathfrak{M}$-compact.

For each ordinal number $a<\mathfrak{M}$ choose a compact space X_a and fix a point $p_a\in X_a$. Let C be the set of all points x in the product space $\prod X_a$ such that $|\{a|x_a\neq p_a\}|\leqq\mathfrak{N}$.

In case each X_a is also a topological group with identity p_a, then C is a topological group.

We next give a technique which can be used, for each $\mathfrak{M}$, to obtain spaces that are $\mathfrak{M}$-compact but not $\mathfrak{M}$-bounded.

LEMMA 4.4. *Let S be an $\mathfrak{M}$-compact space, and suppose that A is a subset of S with $|A|\leqq 2^{\mathfrak{M}}$. Then there is a subset A' of S with $A\subset A'$ and $|A'|\leqq 2^{\mathfrak{M}}$ such that for every filter base $\mathfrak{F}$ on A, if $|\mathfrak{F}|\leqq\mathfrak{M}$, then $A'\cap \text{ad}\,\mathfrak{F}\neq\emptyset$.*

PROOF. Let $\boldsymbol{T}=\{\mathfrak{F}|\mathfrak{F}$ is a filter base on A, $|\mathfrak{F}|\leqq\mathfrak{M}$, and for every $F\in\mathfrak{F}$, $|F|\leqq\mathfrak{M}\}$. Since the number of subsets of A with $\leqq\mathfrak{M}$-points is $\leqq(2^{\mathfrak{M}})^{\mathfrak{M}}=2^{\mathfrak{M}}$, $\boldsymbol{T}$ is a collection of families $\mathfrak{F}$, with $|\mathfrak{F}|\leqq\mathfrak{M}$, constructed by choosing $|\mathfrak{F}|$ elements from a set with $\leqq 2^{\mathfrak{M}}$ elements. Thus $|\boldsymbol{T}|\leqq 2^{\mathfrak{M}}$.

For each $\mathfrak{F}\in\boldsymbol{T}$ choose $p(\mathfrak{F})\in\text{ad}_S\,\mathfrak{F}$, and let $A'=\{p(\mathfrak{F})|\mathfrak{F}\in\boldsymbol{T}\}$. Then $|A'|\leqq 2^{\mathfrak{M}}$. Because S is T_1, each ad$\{\{a\}\}=\{a\}$, so $A\subset A'$.

Let $\mathcal{G}$ be any filter base on A such that $|\mathcal{G}|\leqq\mathfrak{M}$. That $A'\cap\text{ad}\,\mathcal{G}\neq\emptyset$ can be seen as follows. For each $G\in\mathcal{G}$ choose $x(G)\in G$, and let $K=\{x(G)|G\in\mathcal{G}\}$ and $\mathfrak{F}=\mathcal{G}|K$. Since $\mathfrak{F}$ is a filter base and $|\mathcal{G}|$, $|K|\leqq\mathfrak{M}$, we have $\mathfrak{F}\in\boldsymbol{T}$ and so $\emptyset\neq A'\cap\text{ad}\,\mathfrak{F}\subset A'\cap\text{ad}\,\mathcal{G}$.

THEOREM 4.5. *Let S be an $\mathfrak{M}$-compact Hausdorff space containing a subset A such that $|A|=\mathfrak{M}$, and $|\overline{A}|=2^{2^{\mathfrak{M}}}$. Then there is a set $A\subset P\subset S$ such that $|P|\leqq 2^{\mathfrak{M}}$ and P is $\mathfrak{M}$-compact but not $\mathfrak{M}$-bounded. Furthermore, if S is a topological group, then P can also be taken to be a topological group.*

PROOF. We use "′" as in Lemma 4.4. Let $\aleph_a$ be the successor of $\mathfrak{M}$. Put $P_0=A$ and for each ordinal number c define $P_c=[\bigcup\{P_b|b<c\}]'$, and take $P=\bigcup\{P_c|c<\omega_a\}$. Then an inductive argument shows that each $|P_c|$, $c<\omega_a$, is $\leqq 2^{\mathfrak{M}}$, so $|P|\leqq 2^{\mathfrak{M}}$.

To see that P is $\mathfrak{M}$-compact, consider any filter base $\mathfrak{F}$ on P for which $|\mathfrak{F}|\leqq\mathfrak{M}$. Then there is a subset L of P such that $|L|\leqq\mathfrak{M}$ and $\mathfrak{F}|L$ is a filter base. Because $\aleph_a$ is regular, $L\subset P_c$ for some $c<\omega_a$. Thus $\emptyset\neq P_{c+1}\cap\text{ad}\,\mathfrak{F}|L\subset\text{ad}_P\,\mathfrak{F}$.

Next, suppose that there is a compact set $K\subset P$ such that $A\subset K$. Then we have $|K|\leqq|P|\leqq 2^{\mathfrak{M}}$ and $2^{2^{\mathfrak{M}}}=|\overline{A}|\leqq|\overline{K}|$. On the other hand, since S is Hausdorff, it must also be true that $\overline{K}=K$.

In order to see that the last statement of the theorem holds, all one need do is alter the above definition of P by taking each P_c to be the group generated by $[\bigcup\{P_b \mid b<c\}]'$. Then P is the union of a chain of groups and hence is a group.

REMARK 4.6. In [2, Lemma 2.9] Frolík constructs a countably compact completely regular space P such that $|P| = \mathfrak{c}$. For the case $A = \boldsymbol{N}$ and $S = \beta\boldsymbol{N}$, the construction for the second statement in Theorem 4.5 is essentially the same as Frolík's.

THEOREM 4.7. *Let $\mathfrak{M}$ be a regular cardinal. Let S be an $\mathfrak{M}$-compact Hausdorff space containing a subset A such that $|A| = \mathfrak{M}$, and for every $E \subset A$, $|E| = \mathfrak{M} \Rightarrow |\overline{E}| = 2^{2^{\mathfrak{M}}}$. Then there is a set $A \subset P \subset S$ such that $|P| \leqq 2^{\mathfrak{M}}$ and P is $\mathfrak{M}$-compact but not strongly $\mathfrak{M}$-compact. If S is a topological group, then P can be taken to be one also.*

PROOF. Let P be the space constructed in the proof of Theorem 4.5.

Well-order A as $\{a_c \mid c \in \mathfrak{M}\}$, and let $\mathfrak{F}$ be the filter base generated by the sets $\{a_d \mid d \geqq c\}$, $c \in \mathfrak{M}$. Then $|\mathfrak{F}| = \mathfrak{M}$, so if P is strongly $\mathfrak{M}$-compact, there must be a compact set $K \subset P$ such that $\mathfrak{F} \mid K$ is a filter base. Let $D = \{d \in \mathfrak{M} \mid a_d \in K\}$. Since $\mathfrak{M}$ is regular and D is cofinal in $\mathfrak{M}$, $\mathfrak{M} = |D|$. Thus $|K \cap A| = \mathfrak{M}$ and $2^{2^{\mathfrak{M}}} = |[K \cap A]^-| \leqq |\overline{K}| = |K|$ whereas $|P| \leqq 2^{\mathfrak{M}}$.

EXAMPLE 4.8. Let X be any completely regular space containing a dense subset A such that $|A| = \mathfrak{M}$ and $|\beta X| = 2^{2^{\mathfrak{M}}}$. Take S to be βX or any compact group containing βX (e.g. take S to be C^D where C = the circle group and D = the set of continuous mappings of X into C). Then $|\mathrm{Cl}_S A| = |\mathrm{Cl}_{\beta X} A| = 2^{2^{\mathfrak{M}}}$, so the hypothesis of Theorem 4.5 is satisfied. Two examples of this sort are the following:

(i) Let $A = \boldsymbol{Q}$ and $X = \boldsymbol{Q}$ or $\boldsymbol{R}$ (see [4]).

(ii) Let $A = X$ = a discrete space of cardinality $\mathfrak{M}$.

In connection with (ii), recall that if C is any compact zero dimensional space with base $\mathfrak{B}$, then the rule

$$f(c)_B = 1 \quad \text{if } c \in B, \qquad = 0 \quad \text{if } c \notin B,$$

defines a homeomorphism of C into the product space $\{0, 1\}^{\mathfrak{B}}$. Thus βA is embedded in the topological group $S = \{0, 1\}^{2^{\mathfrak{M}}}$.

EXAMPLE 4.9. Let $\mathfrak{M}$ be a regular cardinal and let A and S be as in (ii). Then for every $E \subset A$, $\mathrm{Cl}_{\beta A} E = \beta E$, so $|\mathrm{Cl}_S E| = 2^{2^{|E|}}$. By Theorem 4.7 the space P is $\mathfrak{M}$-compact but not strongly $\mathfrak{M}$-compact.

EXAMPLE 4.10. For each ordinal number $c \in 2^{\mathfrak{M}}$ let X_c be a compact group or Hausdorff space such that $|X_c| \geqq 2$ and X_c has a dense subset of cardinality $\leqq \mathfrak{M}$. If one takes for S the product space $\prod X_c$ and for A any dense subset of S of cardinality $\mathfrak{M}$, then the hypothesis of Theorem 4.5 is satisfied.

As mentioned earlier, we do not know, for $\mathfrak{M} > \aleph_0$, if there exist $\mathfrak{M}$-compact spaces whose product is not $\mathfrak{M}$-compact; however, for regular spaces we can settle the question for certain cardinals $\mathfrak{M}$ and, assuming the Generalized Continuum Hypothesis, for all singular cardinals $\mathfrak{M}$.

THEOREM 4.11. *Let $\mathfrak{M}$ be a singular cardinal such that $2^{\mathfrak{N}} \leqq \mathfrak{M}$ for every $\mathfrak{N} < \mathfrak{M}$, and suppose that X is a regular topological space which is a product of $\mathfrak{M}$-compact spaces. Then X is $\mathfrak{M}$-compact.*

PROOF. It follows from Theorem 2.1 that for every $\mathfrak{N} < \mathfrak{M}$, all factor spaces and, hence, X are $\mathfrak{N}$-bounded. Let $\mathfrak{F} = \{F_m \mid m \in \mathfrak{M}\}$ be a filter base on X. For each $m \in \mathfrak{M}$,

$$G_m = \bigcap \{[F_n]^- \mid n \leqq m\} \neq \emptyset.$$

Let C be any cofinal subset of $\mathfrak{M}$ such that $|C| < \mathfrak{M}$. Then

$$\emptyset \neq \bigcap \{G_c \mid c \in C\} = \operatorname{ad} \mathfrak{F}.$$

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