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ON AN ASYMPTOTIC PROPERTY OF A VOLTERRA INTEGRAL EQUATION

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ABSTRACT. It is proved that if $q(t-s)$ is bounded and $f(t, x)$ is "small," the solutions of the integral equation $x(t)=a(t)+\int_0^t q(t-s)f(s, x(s))ds$ satisfies the conditions $x(t)=h(t)+\rho(t)a(t)$, $\lim_{t\to\infty} a(t)=a$ constant where $\rho(t)$ is a nonsingular diagonal matrix chosen in such a way that $\rho(t)h(t)$ is bounded. The results are extended to the more general integral equation $x(t)=h(t)+\int_0^t F(t, s, x(s))ds$ and contain, in particular, some results on the boundedness, asymptotic behavior and existence of nonoscillatory solution of differential equations.

Consider the system of Volterra integral equations

$$(1) \qquad x(t) = h(t) + \int_0^t q(t-\tau)f(\tau, x(\tau))d\tau + \int_0^t q(t-\tau)g(\tau)d\tau$$

where $h=(h_1, h_2, \cdots, h_n), f=(f_1, f_2, \cdots, f_n), g=(g_1, g_2, \cdots, g_n)$ are column vectors in E^n, $q=(q_{ij})$ is an $n\times n$ matrix and E^n is Euclidean n-dimensional space. In what follows $\lim_{t\to\infty}$ means limit almost everywhere.

We assume the hypotheses

(H_1) $|q(t)|$ and $|g(t)| \in L(0, C]$, $0<C<\infty$,

(H_2) $h(t)$ is continuous for $0\leqq t<\infty$,

(H_3) $f(t, x)$ is continuous for $0\leqq t<\infty$, $|x|<\infty$.

These conditions guarantee the local existence of continuous solutions and continuability of each solution so long as they remain bounded [6, p. 324]. We show here that if f, g and q satisfy some additional hypotheses then the solutions of (1) satsify the condition $x_i(t)=h_i(t)+a_i(t)\rho_i(t)$ with $\lim_{t\to\infty} a_i(t)=a_i$ or $a_i(t)$ bounded, where $\rho_i(t)>0$ is chosen in such a way that $h_i(t)/\rho_i(t)$ is bounded. Some similar results were obtained by Strauss [8] under different hypotheses. We need the following lemma.

LEMMA 1. *Let E be a measurable set of points of any number of dimensions of finite or infinite measure and let $f(t, s)$ be summable in E for*

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values of t in $[0, \infty)$. Assume that there exists a summable nonnegative function $\phi(s)$ such that $|f(t, s)| \leqq \phi(s)$ for almost all values of s in E and all values of t in (t_0, ∞). Then if $\lim_{t\to\infty} f(t, s)$ exists for all (or almost all) values of s in E we have

$$\lim_{t\to\infty} \int_E f(t, s)ds = \int_E \lim_{t\to\infty} f(t, s)ds.$$

For a proof of Lemma 1, see [4, p. 322]. The following lemma which is proved in a more general form in [6, p. 326] gives conditions for the global existence of solutions of (1).

LEMMA 2. *Let $\omega(t, r)$ be continuous in (t, r) for $0 \leqq t < \infty$, $0 \leqq r < \infty$, and nondecreasing in r for each fixed t. For some $b < \infty$ let r_M be the maximum solution of*

$$(2) \qquad \dot{r} = K\omega(t, r), \qquad 0 \leqq t < b, \qquad ' = d/dt, \qquad r(0) = K,$$

and suppose that $|h(t)| \leqq K$, $|q(t)| \leqq K$ and $|f(t, x)| \leqq \omega(t, |x|)$, $0 \leqq t < \infty$, $|x| < \infty$. Then if ϕ is a solution of (1) ϕ can be continued to the right as far as r_M exists and

$$|\phi(t)| \leqq r_M(t), \qquad 0 \leqq t < b.$$

In particular, if $r_M(t)$ exists for $0 \leqq t < \infty$ one has global existence for solutions of (1).

LEMMA 3. *Let $a_i \geqq 0$, $b_i \geqq 0$, $r_i \geqq 0$ and $r = \max_i r_i$, $i = 1, 2, \cdots, n$, if $b_i > 1$ for some i then*

$$\sum_{i=1}^{n} a_i b_i^{r_i} \leqq \left[\sum_{i=1}^{n} a_i\right]\left[\sum_{i=1}^{n} b_i\right]^r.$$

Let $\rho_i(t) > 0$, $i = 1, 2, \cdots, n$, be continuous functions for $0 \leqq t < \infty$ in such a way that $h_i(t)/\rho_i(t)$, $i = 1, 2, \cdots, n$, are bounded for $0 \leqq t < \infty$.

We assume also the hypotheses

(H_4) $|f_i(t, x)| \leqq \sum_{j=1}^{n} \epsilon_{ij}(t)|x_j|^{r_j}$, $i = 1, 2, \cdots, n$, where $\epsilon_{ij}(t)$ is nonnegative, $r_j > 0$, $j = 1, 2, \cdots, n$, $|x| < H \leqq \infty$ and

$$\int^{\infty} \epsilon_{ij}(t)[\rho_j(t)]^{r_j}[\rho_i(t)]^{-1}dt < \infty.$$

(H_5) $\int^{\infty} |g_i(t)| [\rho_i(t)]^{-1}dt < \infty$.

(H_6) $q_{ij}(t-\tau)\rho_j(\tau)/\rho_i(t)$ are bounded for $0 \leqq \tau \leqq t < \infty$, $i, j = 1, 2, \cdots, n$.

(H_6') $q_{ij}(t-\tau)\rho_j(\tau)/\rho_i(t)$ are bounded for $0 \leqq \tau \leqq t < \infty$ and $\lim_{t\to\infty} (q_{ij}(t-\tau)/\rho_i(t))$ exists for almost all values of τ.

THEOREM 1. *Assume with respect to equation* (1) *the hypotheses* (H_4), (H_5) *and* (H_6) *and let r be the maximum of the r_i, $i=1, 2, \cdots, n$. Then*

A. *If $r>1$ there exists $c>0$ depending on r and ϵ_{ij}, $i, j=1, 2, \cdots, n$, such that $|h_i(t)/\rho_i(t)|\leqq c$, $i=1, 2, \cdots, n$, implies that the solutions of* (1) *exist in* $[0, \infty)$ *and satisfy the condition*

$$x_i(t) = h_i(t) + a_i(t)\rho_i(t)$$

where $a_i(t)$ is bounded. If instead of condition (H_6), (1) *satisfies* (H_6') *then* $\lim_{t\to\infty} a_i(t)=a_i$ *constant. Furthermore, if* $\lim_{t\to\infty}(h_i(t)/\rho_i(t))=h_i$ *constant*

$$\lim_{t\to\infty}\frac{x_i(t)}{\rho_i(t)} = b_i \text{ constant.}$$

B. *If $r\leqq 1$, c can be chosen arbitrarily.*

PROOF.

$$x_i(t) = h_i(t) + \int_0^t \sum_{r=1}^n q_{ir}(t-s)f_r(s, x(s))ds + \int_0^t \sum_{j=1}^n q_{ij}(t-s)g_j(s)ds$$

or

$$\begin{aligned}
\frac{|x_i(t)|}{\rho_i(t)} &\leqq k_i + \int_0^t \sum_{r=1}^n \frac{|q_{ir}(t-s)|\,\rho_r(s)}{\rho_i(t)} \sum_{j=1}^n \epsilon_{rj}(s)[\rho_j(s)]^{r_j}\rho_r^{-1}(s)\,\frac{|x_j(s)|^{r_j}ds}{[\rho_j(s)]^{r_j}} \\
&\quad + \int_0^t \sum_{j=1}^n \frac{|q_{ij}(t-s)|\,\rho_j(s)\rho_j(s)^{-1}|}{\rho_i(t)}\,|g_j(s)|\,ds \\
&\leqq k_i + \int_0^t \sum_{r=1}^n c_r \sum_{j=1}^n \epsilon_{rj}(s)[\rho_j(s)]^{r_j}\rho_r(s)^{-1}\frac{|x_j(s)|^{r_j}}{[\rho_j(s)]^{r_j}}\,ds \\
&\quad + \int_0^t \sum_{j=1}^n c_j\,|g_j(s)\rho_j(s)^{-1}|\,ds \\
&\leqq K_i + k\int_0^t \Big(\sum_{rj}\epsilon_{rj}(s)[\rho_j(s)]^{r_j}\rho_r(s)^{-1}\Big)\sum_{i=1}^n \frac{|x_i(s)|^{r_i}}{[\rho_i(s)]^{r_i}}\,ds,
\end{aligned}$$

where $K_i=k_i+\int_0^\infty \sum_{j=1}^n c_j|g_j(s)|\rho_j(s)^{-1}|ds$. If $|x_i(s)|/\rho_i(s)\leqq 1$ for every $i=1, 2, \cdots, n$, $\sum_{i=1}^n (|x_i(s)|^{r_i}/|\rho_i(s)|^{r_i})$ is bounded. If for some i, $|x_i(s)|/\rho_i(s)\geqq 1$, then

$$\sum_{i=1}^n \frac{|x_i(s)|^{r_i}}{[\rho_i(s)]^{r_i}} \leqq n\Big(\sum_{i=1}^n \frac{|x_i(s)|}{\rho_i(s)}\Big)^r$$

by Lemma 3, hence

$$\sum_{i=1}^{n} \frac{|x_i(t)|}{\rho_i(t)} \leqq M + M \int_0^t \left(\sum_{rj} \epsilon_{rj}(s) [\rho_j(s)]^{r_j} \rho_r(s)^{-1} \right) \left(\sum_{i=1}^{n} \frac{|x_i(s)|}{\rho_i(s)} \right)^r ds.$$

By Lemma 2, $\sum_{i=1}^{n} (|x_i(t)|/\rho_i(t)) \leqq z(t)$ where $z(t)$ is the maximal solution of the differential equation

$$\dot{z} = M \left(\sum_{r,j} \epsilon_{rj}(s) [\rho_j(s)]^{r_j} \rho_r(s)^{-1} \right) z^r. \tag{3}$$

The solution of (3) is

$$z(t) = z(0) \exp M \int_0^t \sum_{r,j} \epsilon_{rj}(s) [\rho_j(s)]^{r_j} \rho_r(s)^{-1} ds \quad \text{if } r = 1,$$

$$z(t)^{1-r} = z(0)^{1-r} + (1-r) \exp M \int_0^t \sum_{r,j} \epsilon_{rj}(s) [\rho_j(s)]^{r_j} \rho_r(s)^{-1} ds \quad \text{if } r > 1.$$

Then if $r=1$ all solutions of (3) are bounded in $[0, \infty)$. If $r>1$ choosing $|z(0)| \leqq c$ for c small enough the solutions of (3) are also bounded. Then by Lemma 2 the solutions of (1) exist globally.

Thus in the conditions of hypothesis (H_6), $x_i(t) = h_i(t) + \rho_i(t) a_i(t)$ where

$$a_i(t) = \int_0^t \sum_{r=1}^{n} \frac{q_{ir}(t-s)}{\rho_i(t)} f_r(s, x(s)) ds + \int_0^t \sum_{j=1}^{n} \frac{q_{ij}(t-s)}{\rho_i(t)} g_j(s) ds$$

is bounded and in the conditions of hypothesis (H_6'), by Lemma 1 $\lim a_i(t) = a_i$ constant. The condition of convergence of the integral $\int_0^t \epsilon_{ij}(t) [\rho_j(t)]^{r_j} [\rho_i(t)]^{-1} dt$ in hypothesis (H_4) cannot be improved as is shown by the simple example

$$x_1(t) = 2t + 1 + \int_0^t 2(t-s)(s+1)^{-2} x_1(s) ds,$$

$$x_2(t) = 2 + \int_0^t 2(s+1)^{-2} x_1(s) ds$$

which has the solution $((t+1)^2, 2(t+1))$ which does not satisfy Theorem 1. In this example, $q_{11}=1$, $q_{12}=t-s$, $q_{21}=0$, $q_{22}=1$, $\epsilon_{11}(t)=0$, $\epsilon_{12}(t)=0$, $\epsilon_{22}(t)=0$, $\epsilon_{21}(t)=2(t+1)^2$, $\rho_1(t)=2(t+1)$, $\rho_2(t)=2$.

The example above shows that Theorem 1 is not true even when $\lim_{t\to\infty} \epsilon_{ij}(t) = 0$. Strauss [8] uses a hypothesis of convergence to zero of the $\epsilon_{ij}(t)$ and obtains similar results but uses a different hypotheses for the kernel $q(t-s)$.

REMARK 1. It is quite obvious that Theorem 1 is also true when $q=q(t-s)$ is not of convolution type. Hypothesis (H_4) to (H_6') were motivated by anterior research on differential equations and are general enough to include implicitly Theorem 2.1 of [3], Theorem 1 of [9] and several results on the boundedness and asymptotic behavior of differential equations, see [2, pp. 37 and 42] for references. The hypotheses imposed to the kernel q can be weakened and are better stated in the context of the more general equation

$$x(t) = h(t) + \int_0^t F(t, s, x(s))ds \tag{4}$$

where $F=(F_1, F_2, \cdots, F_n)$ is continuous in t and x and $|F|$ is locally Lebesgue integrable. Theorems on the existence of continuous solutions, continuability of solutions and comparison theorems for (4) can be found in [5], [6] and [7].

We assume also the hypothesis

$$\frac{|F_i(t, s, x(s))|}{\rho_i(t)} \leqq \sum_{j=1}^{m} g_{ij}(s)\,|x_j(s)|^{r_j}, \qquad i = 1, 2, \cdots, n, \quad |x| < H \leqq \infty, \tag{H_7'}$$

where $g_{ij}(s)$ is nonnegative, $r_j \geqq 0$, $j=1, 2, \cdots, n$, and

$$\int^{\infty} g_{ij}(t)[\rho_i(t)]^{r_j}dt < \infty.$$

(H_7) $\lim_{t\to\infty}(F_i(t, s, x(s))/\rho_i(t))$ exist and is finite for almost all values of s and (H_7) is satisfied.

Then conditions A and B of Theorem 1 are satisfied for equation (4), that is, the solutions of (4) satisfy $x_i(t)=h_i(t)+\rho_i(t)a_i(t)$ with $a_i(t)$ bounded in the conditions of hypothesis (H_7) and $\lim_{t\to\infty} a_i(t)=a_i$ in the conditions of hypothesis (H_7'). Still more generally let $\rho=(\rho_i)$ be a diagonal matrix, $\rho_i(t)>0$, $i=1, 2, \cdots, n$, chosen in such a way that $|\rho(t)^{-1}h(t)| \leqq K$. Suppose that $|\rho(t)^{-1}F(t, s, x(s))| \leqq \omega(t, s, |\rho(s)^{-1}x(s)|)$ where $\omega(t, s, r)$ is monotone nondecreasing in r for each fixed pair (t, s). Then if the solutions of the integral equation $y(t)=K+\int_0^t \omega(t, s, y(s))ds$ are bounded, the solutions of (4) satisfy $x(t)=h(t)+\rho(t)a(t)$ with $a(t)$ bounded. If in addition $\lim_{t\to\infty}\rho(t)^{-1}F(t, s, x(s))$ exist and is finite then $\lim_{t\to\infty} a(t)=a$ const.

REMARK 2. The following example shows that without hypothesis (H_7') we can not really guarantee that $\lim_{t\to\infty} a_i(t)$ exist. Consider the linear integral equation

$$x(t) = 1 + \frac{1}{1+t} + \cos\left(\frac{1}{2} - \frac{1}{1+1/(1+t)}\right) - \cos t \int_0^t \frac{1}{(1+s)^2(1+1/(1+s))^3} x(s)ds$$

which has the solution $x(t)=1+1/(1+t)$. $h(t)=1+1/(1+t)-\cos t(1/1+1/(1+t))+\frac{1}{2}\cos t$. We can put $\rho(t)=1+1/(1+t)$. Then

$$\frac{x(t)-h(t)}{\rho(t)} = \frac{\cos t}{1+1/(1+t)} \int_0^t \frac{1}{(s+1)^2(1+1/(1+s))^3} x(s)ds = \frac{\cos t}{1+1/(1+t)}\left[\frac{1}{1+1/(1+t)} - \frac{1}{2}\right]$$

which does not have a limit when $t \to \infty$.

In an analogous manner the integral equation

$$x(t) = 1 + \frac{1}{1+t} - \frac{1}{1+1/(1+t)} + \frac{1}{2} - \int_0^t \frac{1}{(1+s)^2(1+1/(1+s))^3} x(s)ds$$

shows that, in general, we can not have

$$\lim_{t\to\infty} \frac{(x(t)-h(t))}{\rho(t)} = 0.$$

REMARK 3. If $h(t)=$constant in (1), then in the condition of hypothesis (H_6'), $\lim_{t\to\infty} x(t) = C$ constant if $x(t)$ is bounded in $[0, \infty)$. Also the solutions of (1) tend to a point of equilibrium and equation (1) has asymptotic equilibrium in the sense defined by Wintner for differential equations (see [2, p. 43]).

REMARK 4. As a simple application of Theorem 1 consider the equation

$$y^{(n)} + f(t, y^{(1)}, \cdots, y^{(n-1)}) = g(t) \tag{5}$$

which is equivalent to the integral equation

$$y^{(k)}(t) = \theta_0 + \theta_1 t + \cdots + \theta_{n-k-2}t^{n-k-1} + \int_{t_0}^t \frac{(t-s)^{n-k-1}}{(n-k-1)!} g(s)ds - \int_{t_0}^t \frac{(t-s)^{n-k-1}}{(n-k-1)!} f(s, y^{(1)}(s), \cdots, y^{(n)}(s))ds, \tag{6}$$

$$K = 0, 1, \cdots, n-1.$$

If we choose $\rho_k(t)=t^{n-k-1}$, the kernel of (6) satisfies hypotheses (H_1') and (H_6'). If f and g satisfy the hypotheses of Theorem 1 then (5) has a solution satisfying $\lim (y^{(k)}(t)/t^{n-k-1})=A_K$. Since the integrals in (6) are convergent for t_0 sufficiently large, $A_K\neq 0$, $K=1, 2, \cdots, n-1$, and we obtain Theorem 2.1 of [2] with weaker hypotheses which contains Theorem 1 of [9]. Brauer and Wong [1] treat the same problem from a different point of view and obtain the same results in a more general context.

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REFERENCES

1. F. Brauer and J. S. W. Wong, *On asymptotic behavior of perturbed linear systems*, J. Differential Equations 6(1969), 142–153. MR **39** #570.

2. L. Cesari, *Asymptotic behavior and stability problems in ordinary differential equations*, 2nd ed., Academic Press, New York and Springer-Verlag, Berlin, 1963. MR **27** #1661.

3. T. G. Hallam, *Asymptotic behavior of the solutions of an nth order nonhomogeneous ordinary differential equation*, Trans. Amer. Math. Soc. **122** (1966), 177–194. MR **32** #6000.

4. E. W. Hobson, *The theory of functions of a real variable and the theory of Fourier series*. Vol. 2, Dover, New York, 1958. MR **19**, 1166.

5. R. K. Miller, *Nonlinear Volterra integral equations*, Lecture Note Series, Benjamin, New York (to appear).

6. J. A. Nohel, *Some problems in nonlinear Volterra integral equations*, Bull. Amer. Mat. Soc. **68** (1962), 323–329. MR **26** #2838.

7. T. Sato and A. Iwasaki, *Sur l'équation intégrale de Volterra*, Proc. Japan Acad. **31** (1955), 395–398. MR **17**, 751.

8. A. Strauss, *On a perturbed Volterra integral equation*, Arch. Rational Mech. Anal. (to appear).

9. P. Waltman, *On the asymptotic behavior of solutions of a nonlinear equation*, Proc. Amer. Math. Soc. **15** (1964), 918–923. MR **31** #445.

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