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H_2 OF THE COMMUTATOR SUBGROUP OF A KNOT GROUP

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ABSTRACT. A short topological proof is given for the well-known theorem that if G is a knot group and G' its commutator subgroup, then $H_2(G'; Z)=0$.

The purpose of this note is to give a short topological proof of the following well-known theorem [1], [2], [6], [7]:

THEOREM. *If G is a knot group and G' is its commutator subgroup, then* $H_2(G'; \mathbf{Z})=0$.

PROOF. Let S denote the bounded complement of a tamely embedded S^1 in S^3. S is a compact 3-manifold-with-boundary, and is homotopy equivalent to a finite 2-dimensional simplicial complex K. Let $G=\pi_1(K)$. As is well known [5], K is aspherical ($\pi_i(K)=0, i\geqq 2$), hence K is the Eilenberg-MacLane space $K(G, 1)$. Let $\tilde{K}$ denote the infinite cyclic covering space of K; that is, $\pi_1(\tilde{K})=G'$ (the commutator subgroup of G), and $H_1(K; \mathbf{Z})=J(t)$ (the infinite cyclic multiplicative group generated by t) acts on $\tilde{K}$ as the group of simplicial covering translations. $\tilde{K}$ is also aspherical, and is the Eilenberg-MacLane space for G'.

Let Γ denote the rational group ring of $J(t)$. Following [3], [4] we have for all q that the simplicial chain groups $C_q(\tilde{K}; Q)$ are finitely generated free Γ-modules, with generators in 1-1 correspondence with the q-simplexes of K. Since Γ is a principal ideal domain, then $H_q(\tilde{K}; Q)$ is a f.g. Γ-module for all q.

Now collapsing out the infinite cyclic group of covering translations on $\tilde{K}$ yields the orbit space K. Following Milnor [4], this is expressed algebraically by the short exact sequence of chain complexes (as Γ-modules)

$$0 \to C_*(\tilde{K}; Q) \xrightarrow{(t-1)} C_*(\tilde{K}; Q) \to C_*(K; Q) \to 0$$

which yields the long exact sequence of homology

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$$\cdots \to H_3(K;Q) \to H_2(\tilde{K};Q) \xrightarrow{(t-1)} H_2(\tilde{K};Q) \to H_2(K;Q) \to \cdots.$$

Since K is a homology S^1, then $H_2(\tilde{K};Q) \xrightarrow{(t-1)} H_2(\tilde{K};Q)$ is a Γ-isomorphism. $\tilde{K}$ is 2-dimensional, so $H_2(\tilde{K};Q)$ is isomorphic to the submodule of 2-cycles of $C_2(\tilde{K};Q)$. Γ is a PID, so $H_2(\tilde{K};Q)$ is a f.g. free Γ-module.

Now the sequence $0 \to \Gamma \xrightarrow{(t-1)} \Gamma \to Q \to O$ is exact, and the homomorphism $(t-1)$ respects any direct sum splitting for a Γ-module, so $(t-1): H_2(\tilde{K};Q) \to H_2(\tilde{K};Q)$ can be an epimorphism only if $H_2(\tilde{K};Q)=0$. Again, since $\tilde{K}$ is 2-dimensional, $H_2(\tilde{K};\boldsymbol{Z})$ must be free abelian, hence by the Universal Coefficient Theorem $H_2(\tilde{K};\boldsymbol{Z})=0$. Since $H_2(G';\boldsymbol{Z})=H_2(\tilde{K};\boldsymbol{Z})$, the theorem is proved.

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