

H_2 OF THE COMMUTATOR SUBGROUP OF A KNOT GROUP

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ABSTRACT. A short topological proof is given for the well-known theorem that if G is a knot group and G' its commutator subgroup, then $H_2(G'; \mathbf{Z}) = 0$.

The purpose of this note is to give a short topological proof of the following well-known theorem [1], [2], [6], [7]:

THEOREM. *If G is a knot group and G' is its commutator subgroup, then $H_2(G'; \mathbf{Z}) = 0$.*

PROOF. Let S denote the bounded complement of a tamely embedded S^1 in S^3 . S is a compact 3-manifold-with-boundary, and is homotopy equivalent to a finite 2-dimensional simplicial complex K . Let $G = \pi_1(K)$. As is well known [5], K is aspherical ($\pi_i(K) = 0$, $i \geq 2$), hence K is the Eilenberg-MacLane space $K(G, 1)$. Let \tilde{K} denote the infinite cyclic covering space of K ; that is, $\pi_1(\tilde{K}) = G'$ (the commutator subgroup of G), and $H_1(K; \mathbf{Z}) = J(t)$ (the infinite cyclic multiplicative group generated by t) acts on \tilde{K} as the group of simplicial covering translations. \tilde{K} is also aspherical, and is the Eilenberg-MacLane space for G' .

Let Γ denote the rational group ring of $J(t)$. Following [3], [4] we have for all q that the simplicial chain groups $C_q(\tilde{K}; Q)$ are finitely generated free Γ -modules, with generators in 1-1 correspondence with the q -simplexes of K . Since Γ is a principal ideal domain, then $H_q(\tilde{K}; Q)$ is a f.g. Γ -module for all q .

Now collapsing out the infinite cyclic group of covering translations on \tilde{K} yields the orbit space K . Following Milnor [4], this is expressed algebraically by the short exact sequence of chain complexes (as Γ -modules)

$$0 \rightarrow C_*(\tilde{K}; Q) \xrightarrow{(t-1)} C_*(\tilde{K}; Q) \rightarrow C_*(K; Q) \rightarrow 0$$

which yields the long exact sequence of homology

Received by the editors May 24, 1970 and, in revised form, August 14, 1970.

AMS 1970 subject classifications. Primary 55A25, 18H10.

Key words and phrases. Commutator subgroup of a knot group, homology of groups, infinite cyclic covering space.

¹ Partially supported by NSF GP-11943.

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$$\cdots \rightarrow H_2(K; Q) \rightarrow H_2(\tilde{K}; Q) \xrightarrow{(t-1)} H_2(\tilde{K}; Q) \rightarrow H_2(K; Q) \rightarrow \cdots$$

Since K is a homology S^1 , then $H_2(\tilde{K}; Q) \xrightarrow{(t-1)} H_2(\tilde{K}; Q)$ is a Γ -isomorphism. \tilde{K} is 2-dimensional, so $H_2(\tilde{K}; Q)$ is isomorphic to the submodule of 2-cycles of $C_2(\tilde{K}; Q)$. Γ is a PID, so $H_2(\tilde{K}; Q)$ is a f.g. free Γ -module.

Now the sequence $0 \rightarrow \Gamma \xrightarrow{(t-1)} \Gamma \rightarrow Q \rightarrow 0$ is exact, and the homomorphism $(t-1)$ respects any direct sum splitting for a Γ -module, so $(t-1): H_2(\tilde{K}; Q) \rightarrow H_2(\tilde{K}; Q)$ can be an epimorphism only if $H_2(\tilde{K}; Q) = 0$. Again, since \tilde{K} is 2-dimensional, $H_2(\tilde{K}; \mathbf{Z})$ must be free abelian, hence by the Universal Coefficient Theorem $H_2(\tilde{K}; \mathbf{Z}) = 0$. Since $H_2(G'; \mathbf{Z}) = H_2(\tilde{K}; \mathbf{Z})$, the theorem is proved.

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