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MEAN CONVERGENCE AND COMPACT SUBSETS OF L_1

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ABSTRACT. It is shown that in the usual criterion for w compactness of a set K in $L_1(\mu)$ the explicit assumption that K is bounded follows from the other assumptions which are usually made if μ is nonatomic.

We establish a result connecting the concepts of equicontinuity from above at ϕ and uniform absolute continuity for a collection of measures. We draw from this result sharper forms of known theorems on mean convergence and compact subsets of L_1.

In Paul Halmos' book *Measure theory*, there appears the following theorem (Theorem C, p. 108):

THEOREM. *A sequence $\{f_n\}$ of integrable functions converges in the mean to the integrable function f if and only if $\{f_n\}$ converges in measure to f and the indefinite integrals of $\{f_n\}$, $n=1, 2, \cdots$, are uniformly absolutely continuous and equicontinuous from above at ϕ. (In the "if" direction the function f is assumed measurable and it is part of the conclusion that f is integrable.)*

We will prove the following sharpening of this result in the "if" direction.

THEOREM 2. *If a sequence $\{f_n\}$ of integrable functions converges in measure to a measurable function f and the indefinite integrals of $\{f_n\}$, $n=1, 2, \cdots$, are equicontinuous from above at ϕ, then f is integrable and $\{f_n\}$ converges in the mean to f.*

Dunford and Schwartz prove in their book *Linear operators*. Part I the following theorem (Theorem 9, p. 292):

THEOREM. *A subset K of $L_1(S, \Sigma, \mu)$ is weakly sequentially compact if and only if it is bounded and the indefinite integrals of the f in K are equicontinuous from above at ϕ.*

This can be sharpened in the "if" direction in the special case where μ is atom free.

THEOREM 3. *If μ is atom free then a sufficient condition for a subset*

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$K \subset L_1(S, \Sigma, \mu)$ to be weakly sequentially compact is that the indefinite integrals of the f in K are equicontinuous from above at ϕ.

Theorems 2 and 3 follow from the quoted theorems and the fact that the omitted hypothesis in each case can be shown to be a consequence of the remaining hypotheses. In fact this is the mode in which we will establish these results.

PROOFS. Let X be a set and S a σ-ring of subsets of X. We recall that a collection C of finite real or complex valued measures on S are said to be equicontinuous from above at ϕ, provided each decreasing sequence $\{E_n\}$ in S with void intersection the limit $\lim_n v(E_n)=0$ is uniform for $v \in C$. (Dunford and Schwartz in [1] refer to this notion as "the countable additivity of the measures being uniform with respect to $v \in C$".)

Our first theorem is the central result of this paper from which the other results follow.

THEOREM 1. *Let X be a set, S a σ-ring subsets of X and μ a non-negative real valued measure defined on S. If C is a collection of finite real valued signed measures defined on S whose members are individually absolutely continuous with respect to μ and equicontinuous from above at ϕ then the members of C are uniformly absolutely continuous with respect to μ.*

PROOF. We will prove this by contradiction. We assume that the members of C are not uniformly absolutely continuous with respect to μ. Under this hypothesis we can find a $\Delta > 0$ and construct a sequence of sets $\{E_n\}$ and a corresponding sequence of measures $\{V_n\} \subset C$ such that

$$|V_n(E_n)| > \Delta, \qquad \left|V_n\left(\bigcup_{j=n+1}^{\infty} E_j - E_n\right)\right| \leqq \tfrac{1}{2}\Delta,$$

and

$$\lim_{n\to\infty} \mu\left(\bigcup_{j=n}^{\infty} E_j\right) = 0.$$

Then we will form the decreasing sequence $F_n \downarrow \phi$ by setting

$$F_n = \bigcup_{j=n}^{\infty} E_j - \bigcap_{j=1}^{\infty} \bigcup_{l=j}^{\infty} E_l.$$

The sequence F_n will contradict the equicontinuity from above at ϕ.

Now we proceed with the actual proof. Since each $V \in C$ is absolutely continuous with respect to μ, for every $\epsilon > 0$ and $V \in C$, there exists a $\delta_V(\epsilon)$ such that $E \in S$ and $\mu(E) \leqq \delta_V(\epsilon)$ implies $|V(E)| \leqq \epsilon$. Now assume that the members of C are not uniformly absolutely continuous with respect to μ. Then there is a $\Delta > 0$ such that for each $d > 0$ there exists a $V^d \in C$ and $E^d \in S$ such that

$$\mu(E^d) < d \quad \text{and} \quad |V^d(E^d)| > \Delta. \tag{1}$$

Now define d_n, V_n and E_n recursively by

$$d_1 = \tfrac{1}{2}, \qquad V_1 = V^{d_1}, \qquad E_1 = E^{d_1},$$
$$d_{n+1} = \min(d_n/2,\ \delta_{V_n}(\Delta/2)/2),$$
$$V_{n+1} = V^{d_{n+1}}, \qquad E_{n+1} = E^{d_{n+1}}.$$

Next set $F_n = \bigcup_{j=n}^{\infty} E_j - \bigcap_{j=1}^{\infty} \bigcup_{l=j}^{\infty} E_l$. We claim that the F_n contradict the equicontinuity from above at ϕ of the members of C. Clearly by (1)

$$|V_n(E_n)| = |V^{d_n}(E^{d_n})| > \Delta \quad \text{for all } n. \tag{2}$$

Also observe

$$d_j \leqq (\tfrac{1}{2})^{j-n} d_n \quad \text{for } j \geqq n \geqq 1$$

and thus

$$d_j \leqq (\tfrac{1}{2})^{j-n-1} d_{n+1} \leqq (\tfrac{1}{2})^{j-n} \delta_{V_n}(\Delta/2) \quad \text{for } j > n \geqq 1.$$

Hence

$$\sum_{j=n+1}^{\infty} d_j \leqq \delta_{V_n}\left(\frac{\Delta}{2}\right)$$

and

$$\sum_{j=n+1}^{\infty} d_j \leqq d_n \leqq (\tfrac{1}{2})^{n-1} d_1 = (\tfrac{1}{2})^n \quad \text{for } n \geqq 1.$$

Therefore, using (1) we have

$$\begin{aligned} \mu\left(\bigcup_{j=n+1}^{\infty} E_j\right) &\leqq \sum_{j=n+1}^{\infty} \mu(E_j) = \sum_{j=n+1}^{\infty} \mu(E^{d_j}) \\ &\leqq \sum_{j=n+1} d_j \leqq \min(\delta_{V_n}(\Delta/2), (\tfrac{1}{2})^n). \end{aligned} \tag{3}$$

It follows that

$$\mu\left(\bigcup_{j=n+1}^{\infty} E_j - E_n\right) \leqq \mu\left(\bigcup_{j=n+1}^{\infty} E_j\right) \leqq \delta_{V_n}\left(\frac{\Delta}{2}\right)$$

and consequently

$$\left| V_n\left(\bigcup_{j=n+1}^{\infty} E_j - E_n \right) \right| \leqq \frac{\Delta}{2} \cdot \tag{4}$$

It also follows from (3) that

$$\lim_{n\to\infty} \mu\left(\bigcup_{j=n}^{\infty} E_j \right) = 0. \tag{5}$$

Since $\{\bigcup_{l=j}^{\infty} E_l\}_{j=1,2,\ldots}$ is a decreasing sequence and $\mu(\bigcup_{l=2}^{\infty} E_l) \leqq \frac{1}{2} < \infty$ by (3) we have from (5)

$$\mu\left(\bigcap_{j=1}^{\infty} \bigcup_{l=j}^{\infty} E_l \right) = \lim_{j\to\infty} \mu\left(\bigcup_{l=j}^{\infty} E_l \right) = 0.$$

Thus by the absolute continuity of the members of C with respect to μ it follows that

$$V_n\left(\bigcap_{j=1}^{\infty} \bigcup_{l=j}^{\infty} E_l \right) = 0 \quad \text{for all } n \geqq 1.$$

It now follows from (2), (4) and (6) that

$$\begin{aligned}
| V_n(F_n) | &= \left| V_n\left(\bigcup_{j=n}^{\infty} E_j - \bigcap_{j=1}^{\infty} \bigcup_{l=j}^{\infty} E_l \right) \right| \\
&= \left| V_n\left(\bigcup_{j=n}^{\infty} E_j \right) - V_n\left(\bigcap_{j=1}^{\infty} \bigcup_{l=j}^{\infty} E_l \right) \right| = \left| V_n\left(\bigcup_{j=n}^{\infty} E_j \right) \right| \\
&= \left| V_n\left(E_n \cup \left(\bigcup_{j=n+1}^{\infty} E_j - E_n \right) \right) \right| \\
&= \left| V_n(E_n) + V_n\left(\bigcup_{j=n+1}^{\infty} E_j - E_n \right) \right| \\
&\geqq | V_n(E_n) | - \left| V_n\left(\bigcup_{j=n+1}^{\infty} E_j - E_n \right) \right| \\
&\geqq \Delta - \tfrac{1}{2}\Delta = \tfrac{1}{2}\Delta.
\end{aligned}$$

But

$$\begin{aligned}
\bigcap_{n=1}^{\infty} F_n &= \bigcap_{n=1}^{\infty} \left(\bigcup_{j=n}^{\infty} E_j - \bigcap_{j=1}^{\infty} \bigcup_{l=j}^{\infty} E_l \right) \\
&= \bigcap_{n=1}^{\infty} = \bigcup_{j=1}^{\infty} E_j - \bigcap_{j=1}^{\infty} \bigcup_{l=j}^{\infty} E_l = \phi
\end{aligned}$$

and clearly $F_{n+1} \subset F_n$. Thus $\{F_n\}$ is a decreasing sequence of sets with empty intersection for which $V(F_n)$ does not converge to 0 uniformly in $V \in C$. This contradicts the assumption of equicontinuity from above at ϕ for C. Therefore the members of C are uniformly absolutely continuous with respect to μ as we wished to prove. Q.E.D.

We note here that Theorem 1 holds for collections C of finite complex valued signed measures. Indeed this follows immediately from Theorem 1 by considering the real and imaginary parts of such measures.

PROOF OF THEOREM 2. Theorem 2 follows directly from Theorem 1 and the theorem quoted above from Halmos [2]. Q.E.D.

Now using the notation of Dunford and Schwartz [1] we have

PROOF OF THEOREM 3. Without loss of generality μ may be assumed to be finite. To see this we note first that μ may be assumed to be σ-finite since only sequences of measurable functions are involved. Secondly, every σ-finite measure is equivalent to a finite measure. Now Theorem 3 follows from Theorem 1, the theorem quoted above from Dunford and Schwartz [1], and a theorem of Saks (Lemma IV, 9.7 of [1]).

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