PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, No. 1, April 1971

ON SPECIAL GENERATORS FOR PROPERLY INFINITE VON NEUMANN ALGEBRAS[1]

W. R. WOGEN

ABSTRACT. It is known that every properly infinite von Neumann algebra $\mathcal{A}$ on a separable Hilbert space has a single generator. We show in this paper that a generator for $\mathcal{A}$ may be chosen from some special classes of operators. In particular each of the following classes of operators contains a generator for $\mathcal{A}$: the hyponormals, the nilpotents, the transcendental quasinilpotents, and the unimodular contractions. We also show that a generator for $\mathcal{A}$ may be chosen with arbitrarily prescribed spectrum.

Introduction. Let $\mathcal{H}$ be a separable complex Hilbert space. A von Neumann algebra $\mathcal{A}$ is *properly infinite* if $\mathcal{A}$ contains no finite projections in its center. If $\mathcal{S}$ is an algebra of operators, $\mathcal{S}'$ denotes the commutant of $\mathcal{S}$. For $2 \leqq n \leqq \aleph_0$, $M_n(\mathcal{S})$ denotes the algebra of $n \times n$ matrices over $\mathcal{S}$ which act boundedly on $\sum_{i=1}^{n} \oplus \mathcal{H}$. Let $\mathcal{R}(A, B, \cdots)$ denote the von Neumann algebra generated by the set $\{A, B, \cdots\}$ of operators. The reader is referred to [2] and [6] as references on von Neumann algebras.

Throughout this paper we will need the well-known fact that if $\mathcal{A}$ is a properly infinite von Neumann algebra, then $\mathcal{A}$ is *-isomorphic to $M_n(\mathcal{A})$ for $1 \leqq n \leqq \aleph_0$ (cf. [6, Corollary 14]). Also, it is known that a *-isomorphism of a von Neumann algebra $\mathcal{A}$ onto a von Neumann algebra $\mathcal{B}$ carries a generator of $\mathcal{A}$ onto a generator of $\mathcal{B}$ (cf. [6, p. 68]).

It was shown in [7] that if $\mathcal{A}$ is a properly infinite von Neumann algebra, then $\mathcal{A}$ has a single generator. It then follows from [3, Lemma 1] that $\mathcal{A}$ has a partially isometric generator. In this note we will construct some other generators for $\mathcal{A}$.

The author wishes to thank Professor D. Topping and Professor J. Stampfli for many helpful conversations concerning the material presented here.

Results. We begin with a lemma.

Received by the editors May 26, 1970.

AMS 1969 *subject classifications.* Primary 4665, 4710.

Key words and phrases. Generator, properly infinite von Neumann algebra, hyponormal, transcendental quasinilpotent, unimodular contraction.

[1] This paper is part of the author's doctoral dissertation written under the direction of Professor D. Topping.

LEMMA 1. *Let* $3 \leqq n \leqq \aleph_0$, *and let* $\{a_k\}_{k=1}^n$ *and* $\{\lambda_k\}_{k=1}^n$ *be bounded sequences of complex numbers with* $a_k \neq 0$ $\forall k$. *Let* $\mathcal{A}$ *be a von Neumann algebra with* $\mathcal{A} = \mathcal{R}(A)$, *where* A *satisfies* $\|(\lambda_1 - \lambda_2)A\| < |a_1 a_2|$. *Define* $B = (B_{i,j})_{i,j=1}^n \in M_n(\mathcal{A})$ *by* $B_{i,i} = \lambda_i I$, $B_{i+1,i} = a_i I$, $B_{3,1} = A$, *and* $B_{i,j} = 0$ *otherwise. Then* $\mathcal{R}(B) = M_n(\mathcal{A})$.

PROOF. We will show that $\mathcal{R}(B)' = \{(\delta_{i,j} D)_{i,j=1}^n : D \in \mathcal{A}'\}$. It then follows easily that $\mathcal{R}(B) = \mathcal{R}(B)'' = M_n(\mathcal{A})$. Let $C = (C_{i,j})_{i,j=1}^n \in \mathcal{R}(B)'$ with C selfadjoint (i.e., $C_{i,j} = C^*_{j,i}$). Then $BC = (E_{i,j})_{i,j=1}^n$, where

$$E_{1,j} = \lambda_1 C_{1,j}, \qquad E_{3,j} = AC_{1,j} + a_2 C_{2,j} + \lambda_3 C_{3,j},$$

and for $i \neq 1, 3$,

$$E_{i,j} = a_{i-1} C_{i-1,j} + \lambda_i C_{i,j}.$$

$CB = (F_{i,j})_{i,j=1}^n$, where

$$F_{i,1} = \lambda_1 C_{i,1} + a_1 C_{i,2} + C_{i,3} A,$$

and $\forall j \neq 1$,

$$F_{i,j} = \lambda_j C_{i,j} + a_j C_{i,j+1}.$$

By assumption, $BC = CB$. Since $E_{1,1} = F_{1,1}$ and $E_{1,2} = F_{1,2}$, we have $a_1 C_{1,2} + C_{1,3} A = 0$ and $a_2 C_{1,3} = (\lambda_1 - \lambda_2) C_{1,2}$. It follows that $C_{1,3}(a_1 a_2 I + (\lambda_1 - \lambda_2) A) = 0$. But $\|(\lambda_1 - \lambda_2)A\| < |a_1 a_2|$ so $a_1 a_2 I + (\lambda_1 - \lambda_2) A$ is invertible, and $C_{1,3} = 0$. Thus also $C_{1,2} = 0$. Now $E_{1,3} = F_{1,3}$ implies $C_{1,4} = 0$. Proceeding in this way we get $C_{1,k} = 0$ for $k \geqq 2$. Now if we compare $E_{2,j}$ and $F_{2,j}$, we see by similar arguments that $C_{2,k} = 0$ for $k \geqq 3$. Continuing in this way we get $C_{i,j} = 0$ $\forall i < j$. $C_{i,j} = C^*_{j,i}$, so $C_{i,j} = 0$ $\forall i \neq j$.

Since $E_{i+1,i} = F_{i+1,i}$, we have $C_{1,1} = C_{i,i}$ for $i \geqq 1$. Finally, since $E_{3,1} = F_{3,1}$, we have $AC_{1,1} = C_{3,3} A = C_{1,1} A$. $C_{1,1}$ is selfadjoint, so $C_{1,1} \in \mathcal{R}(A)' = \mathcal{A}'$. It follows that $\mathcal{R}(B)' = \{(\delta_{i,j} D)_{i,j=1}^n : D \in \mathcal{A}'\}$ as asserted.

DEFINITION 1. An operator A is *hyponormal* if $A^*A - AA^* \geqq 0$.

Note that hyponormality is invariant under *-isomorphism.

THEOREM 1. *If* $\mathcal{A}$ *is a properly infinite von Neumann algebra on a separable Hilbert space, then* $\mathcal{A}$ *has a hyponormal generator.*

PROOF. Choose $A \in \mathcal{A}$ such that $\mathcal{R}(A) = \mathcal{A}$ and $\|A\| \leqq 1/2$. Let $B = (B_{i,j})_{i,j=1}^\infty \in M_\infty(\mathcal{A})$ be defined by $B_{2,1} = I$, $B_{3,2} = 2I$, $B_{i+1,i} = 3I$ for $i \geqq 3$, $B_{3,1} = A$, and $B_{i,j} = 0$ otherwise. Then by Lemma 1, $\mathcal{R}(B) = M_\infty(\mathcal{A})$. We assert that B is hyponormal. In fact, to show $B^*B - BB^* \geqq 0$, it suffices to show that the 3 by 3 matrix

$$\begin{bmatrix} I + A^*A & 2A^* & 0 \\ 2A & 3I & -A^* \\ 0 & -A & 5I - AA^* \end{bmatrix}$$

is positive. This is a routine computation. Since $\mathcal{A}$ is *-isomorphic to $M_\infty(\mathcal{A})$, it follows that $\mathcal{A}$ has a hyponormal generator.

This theorem shows in particular that there exist hyponormal operators A such that $\mathcal{R}(A)$ is not type I.

We mention that H. Behncke has recently shown [1] that Theorem 1 is true with "hyponormal" replaced by "subnormal".

REMARK 1. If A is hyponormal and $\mathcal{R}(A)$ is finite, then A is normal and $\mathcal{R}(A)$ is abelian. (This holds because if $\mathcal{R}(A)$ is finite, then there is a unique center valued trace function τ on $\mathcal{R}(A)$ (cf. [2, Chapter III, §4]) satisfying, in particular, (1) $\tau(CB-BC)=0\ \forall B,\ C\in\mathcal{R}(A)$, and (2) if $P\geqq 0$ and $\tau(P)=0$, then $P=0$. Hence if $A^*A-AA^*\geqq 0$, then $(A^*A-AA^*)=0$, so $A^*A-AA^*=0$, and A is normal.) It follows that if A is hyponormal, then $\mathcal{R}(A)$ is of the form $\mathcal{A}\oplus\mathcal{B}$, where $\mathcal{A}$ is abelian and $\mathcal{B}$ is properly infinite.

DEFINITION 2. An operator A is *quasinilpotent* if $\lim_{n\to\infty}\|A^n\|^{1/n}=0$. A is *nilpotent of index n* if $A^n=0$ and $A^{n-1}\neq 0$. A is *transcendental quasinilpotent* if A is quasinilpotent but not nilpotent.

REMARK 2. Topping has shown [5, Theorem 3] that a properly infinite von Neumann algebra is linearly spanned by its transcendental quasinilpotents.

THEOREM 2. *Let $\mathcal{A}$ be a properly infinite von Neumann algebra on a separable Hilbert space. Then $\mathcal{A}$ has a transcendental quasinilpotent generator.*

PROOF. Let $\{a_n\}_{n=1}^\infty$ be a sequence of nonzero complex numbers with $\lim_{n\to\infty} a_n=0$. Let A be a generator of $\mathcal{A}$. Define $Q=(Q_{i,j})_{i,j=1}^\infty\in M_\infty(\mathcal{A})$ by $Q_{i+1,i}=a_iI$, $Q_{i,j}=0$ otherwise. Then Q is a weighted shift, and it is easy to show that Q is quasinilpotent. Let $N=(N_{i,j})_{i,j=1}^\infty\in M_\infty(\mathcal{A})$ be defined by $N_{3,1}=A$, $N_{i,j}=0$ otherwise. Let $B=Q+N$. Then $\mathcal{R}(B)=M_\infty(\mathcal{A})$ by Lemma 1. We claim that B is a transcendental quasinilpotent. In fact, a computation shows that $B^n=(Q+N)^n=Q^n+Q^{n-1}N$. Thus $\|B^n\|^{1/n}=\|Q^n+Q^{n-1}N\|^{1/n}\leqq\|Q^{n-1}\|^{1/n}\|Q+N\|^{1/n}\to 0$ as $n\to\infty$. Now note that transcendental quasinilpotence is a *-isomorphism invariant and that $\mathcal{A}$ is *-isomorphic to $M_\infty(\mathcal{A})$. It follows that $\mathcal{A}$ has a transcendental quasinilpotent generator.

THEOREM 3. *If $n\geqq 3$ and $\mathcal{A}$ is a separably acting properly infinite von Neumann algebra, then $\mathcal{A}$ has a nilpotent generator of index n.*

PROOF. If $\mathcal{A}=\mathcal{R}(A)$, let $B=(B_{i,j})_{i,j=1}^{n}\in M_n(\mathcal{A})$ be defined by $B_{i+1,i}=I$, $B_{3,1}=A$, and $B_{i,j}=0$ otherwise. Then $\mathcal{R}(B)=M_n(\mathcal{A})$ by Lemma 1, and B is nilpotent of index n. $\mathcal{A}$ is *-isomorphic to $M_n(\mathcal{A})$, so the theorem follows. (For $n=3$, this theorem is due to Pearcy and Ringrose (unpublished).)

Let $\sigma(A)$ denote the spectrum of A. The next theorem asserts that a properly infinite von Neumann algebra has generators with arbitrarily prescribed spectrum. More precisely,

THEOREM 4. *Let $\mathcal{A}$ be a properly infinite von Neumann algebra on a separable Hilbert space $\mathcal{H}$ and let K be a nonempty compact subset of the complex plane. Then there is an operator B in $\mathcal{A}$ such that $\mathcal{R}(B)=\mathcal{A}$ and $\sigma(B)=K$.*

PROOF. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of complex numbers such that $a_k\neq 0$ $\forall k$ and $\lim_{k\to\infty} a_k=0$. Let $\mathcal{S}$ be a countable dense subset of K. Form a sequence $\{\lambda_k\}_{k=1}^{\infty}$ from $\mathcal{S}$ such that each element of $\mathcal{S}$ occurs countably many times in the sequence. Choose $A\in\mathcal{A}$ with $\mathcal{R}(A)=\mathcal{A}$ and $\|(\lambda_1-\lambda_2)A\|<|a_1a_2|$.

Let $N=(\delta_{i,j}\lambda_iI)_{i,j=1}^{\infty}\in M_\infty(\mathcal{A})$. Let $Q=(Q_{i,j})_{i,j=1}^{\infty}\in M_\infty(\mathcal{A})$ be defined by $Q_{i+1,i}=a_iI$, $Q_{3,1}=A$, and $Q_{i,j}=0$ otherwise. Write $B=Q+N$. Then B is an operator on $\sum_{k=1}^{\infty}\oplus\mathcal{H}_k$, where $\mathcal{H}_k=\mathcal{H}$ $\forall k$. By Lemma 1, $\mathcal{R}(B)=M_\infty(\mathcal{A})$. We will show that $\sigma(B)=K$.

Notice that $\lambda\in\sigma(B)\Leftrightarrow 0\in\sigma(B-\lambda)$, and that the matrix $B-\lambda$ has the same form as B. Hence it suffices to show that $0\in\sigma(B)$ $\Leftrightarrow 0\in K=\{\lambda_k\}^-$.

Suppose first that $0\in\{\lambda_k\}^-$. Then there is a subsequence $\{\lambda_{k_j}\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty}\lambda_{k_j}=0$. Choose $x_{k_j}\in\mathcal{H}_{k_j}$ with $\|x_{k_j}\|=1$, and identify x_{k_j} with the vector $(0, 0, \cdots 0, x_{k_j}, 0, \cdots)\in\sum_{k=1}^{\infty}\oplus\mathcal{H}_k$ whose only nonzero entry is in the k_jth position. Then $\|Bx_{k_j}\|\leqq\|\lambda_{k_j}x_{k_j}\|+\|a_{k_j}x_{k_j}\|$ $=|\lambda_{k_j}|+|a_{k_j}|$ for $k_j>1$. But $\lim_{j\to\infty}a_{k_j}=\lim_{j\to\infty}\lambda_{k_j}=0$, so that $\|Bx_{k_j}\|\to 0$ as $j\to\infty$. Thus $0\in\sigma(B)$. (This argument actually shows that 0 is in the approximate point spectrum of B.)

Now suppose that $0\notin\{\lambda_k\}^-$. Then N is invertible. In fact, N^{-1} $=(\delta_{i,j}\lambda_i^{-1}I)_{i,j=1}^{\infty}$. Computing $N^{-1}Q$, we find that $N^{-1}Q=(C_{i,j})_{i,j=1}^{\infty}$, where $C_{i+1,i}=\lambda_{i+1}^{-1}a_iI$, $C_{3,1}=\lambda_3^{-1}A$, and $C_{i,j}=0$ otherwise. Then $N^{-1}Q$ is quasinilpotent by the proof of Theorem 2, since $\lim_{k\to\infty}\lambda_{k+1}^{-1}a_k=0$. Thus $I+N^{-1}Q$ is invertible. But then $B=N+Q=N(I+N^{-1}Q)$ is the product of invertible operators, so B is invertible, i.e., $0\notin\sigma(B)$. The theorem now follows since $\mathcal{A}$ is *-isomorphic to $M_\infty(\mathcal{A})$.

DEFINITION 3. An operator A is a *unimodular contraction* if $\|A\|\leqq 1$ and $\sigma(A)\subset\{z:|z|=1\}$.

Note that the image under a *-isomorphism of a unimodular contraction is a unimodular contraction. In [4], Russo poses the question: "Do there exist unimodular contractions of type II_∞ and III?" The following theorem answers this question in the affirmative.

THEOREM 5. *If $\mathcal{A}$ is a properly infinite von Neumann algebra on a separable Hilbert space $\mathcal{H}$, then $\mathcal{A}$ has a unimodular contractive generator.*

We first prove a lemma.

LEMMA 2. *Let $\{a_k\}_{k=1}^{\infty}$ be an increasing sequence of positive numbers such that $\lim_{k\to\infty} a_k = 1$. Let the operator $T=(b_{i,j})_{i,j=-\infty}^{\infty}$ on $l^2(\mathbf{Z})$ be defined by $b_{i,i-1}=1$ for $i\leqq 0$, $b_{i,i-1}=a_i$ for $i>0$, and $b_{i,j}=0$ otherwise. Then B is a unimodular contraction.*

PROOF. T is a two-sided weighted shift. Obviously T is a contraction. But an easy computation shows that $\|T^{-n}\| = 1/a_1a_2 \cdots a_n$. Then

$$\lim_{n\to\infty} \|T^{-n}\|^{1/n} = \lim_{n\to\infty} (1/a_1a_2 \cdots a_n)^{1/n} = 1,$$

since $\lim_{n\to\infty} a_n = 1$. Thus the spectral radius of T^{-1} is 1, and it follows that $\sigma(T) \subset \{z: |z| = 1\}$.

PROOF OF THEOREM 5. By Theorem 3, we can choose $A\in\mathcal{A}$ with $\mathcal{R}(A)=\mathcal{A}$ and A nilpotent. Moreover, we may suppose $\|A\|\leqq 1/2$. Let $a_n = n/(n+1)$ for $n=1, 2, \cdots$. Define $Q=(Q_{i,j})_{i,j=-\infty}^{\infty} \in M_\infty(\mathcal{A})$ by $Q_{2,0}=A$ and $Q_{i,j}=0$ otherwise. Define $T=(T_{i,j})_{i,j=-\infty}^{\infty}\in M_\infty(\mathcal{A})$ by $T_{i,i-1}=I$ for $i\leqq 0$, $T_{i,i-1}=a_iI$ for $i>0$, and $T_{i,j}=0$ otherwise. Let $B=T+Q$. We claim that B is a unimodular contractive generator for $M_\infty(\mathcal{A})$.

We show first that B is a contraction. Recall that $\|A\|\leqq 1/2$, $a_1=1/2$, and $a_2=2/3$. Let $x=(x_n)_{n=-\infty}^{\infty} \in \sum_{n=-\infty}^{\infty} \oplus \mathcal{H}$. Then

$$\begin{aligned}\|Bx\|^2 &= \sum_{-\infty}^{-1} \|x_n\|^2 + \|a_1x_0\|^2 + \|Ax_0 + a_2x_1\|^2 + \sum_{2}^{\infty} \|a_{n+1}x_n\|^2 \\ &\leqq \sum_{-\infty}^{-1} \|x_n\|^2 + \|a_1x_0\|^2 + 2(\|Ax_0\|^2 + \|a_2x_1\|^2) + \sum_{2}^{\infty} \|x_n\|^2 \\ &\leqq \sum_{-\infty}^{-1} \|x_n\|^2 + ((1/2)^2 + 2(1/2)^2)\|x_0\|^2 + 2(2/3)^2\|x_1\|^2 + \sum_{2}^{\infty} \|x_n\|^2 \\ &\leqq \|x\|^2.\end{aligned}$$

Thus B is a contraction.

Next we show that $\sigma(B)\subset\{z:|z|\leqq 1\}$. It suffices to show that if $|\lambda|<1$, then $\lambda\notin\sigma(B)$. Let $|\lambda|<1$. By Lemma 2, T is a unimodular contraction. Thus $T-\lambda I$ is invertible. $B=T+Q$, so $B-\lambda I=(T-\lambda I)+Q=(T-\lambda I)[I+(T-\lambda I)^{-1}Q]$. Now simple matrix multiplication shows that $(T-\lambda I)^{-1}Q$ is nilpotent. (In fact if $A^n=0$ and $C\in M_\infty(CI)$, then $(CQ)^n=0$.) Hence $I+(T-\lambda I)^{-1}Q$ is invertible. Since $B-\lambda I$ is a product of invertible operators, $B-\lambda I$ is invertible and $\lambda\notin\sigma(B)$.

Finally we sketch a proof that $\mathcal{R}(B)=M_\infty(\mathfrak{A})$. As in Lemma 1, it suffices to show that $\mathcal{R}(B)'=\{(\delta_{i,j}D)_{i,j=-\infty}^{\infty}:D\in\mathfrak{A}'\}$. Let $C=(C_{i,j})_{i,j=-\infty}^{\infty}\in\mathcal{R}(B)'$ with C selfadjoint. Then $BC=(E_{i,j})_{i,j=-\infty}^{\infty}$, where

$$E_{2,j}=AC_{0,j}+a_2C_{1,j},$$

$$E_{i,j}=C_{i-1,j}\qquad\text{for } i\leqq 0,$$

and

$$E_{i,j}=a_iC_{i-1,j}\qquad\text{for } i>0,\ i\neq 2.$$

$CB=(F_{i,j})_{i,j=-\infty}^{\infty}$, where

$$F_{i,0}=a_iC_{i,1}+C_{i,2}A,$$

$$F_{i,j}=C_{i,j+1}\qquad\text{for } j<0,$$

and

$$F_{i,j}=a_{j+1}C_{i,j+1}\qquad\text{for } j>0.$$

We are assuming that $BC=CB$ and that $C_{i,j}=C_{j,i}^*$.

Let $n\geqq 3$. Since $E_{0,n-1}=F_{0,n-1}$ and $E_{n,-1}=F_{n,-1}$, we find that $a_nC_{0,n}=C_{-1,n-1}$ and $a_nC_{-1,n-1}^*=C_{0,n}^*$. But $a_n\neq 1$, so $C_{0,n}=C_{-1,n-1}=0$. Fix $n\geqq 3$. Comparing $E_{k,k+n-1}$ and $F_{k,k+n-1}$, we find that $C_{k,k+n}=0$ $\forall k$. But $E_{2,3}=F_{2,3}$ and $E_{4,1}=F_{4,1}$, so that $a_2C_{1,3}=a_4C_{2,4}$ and $a_2C_{2,4}^*=a_4C_{1,3}^*$. Since $a_2\neq a_4$, we must have $C_{1,3}=C_{2,4}=0$. It follows that $C_{k,k+2}=0$ $\forall k$. Similarly, since $E_{2,2}=F_{2,2}$ and $E_{3,1}=F_{3,1}$, we get $a_2C_{1,2}=a_3C_{2,3}$ and $a_3C_{1,2}^*=a_2C_{2,3}^*$. Hence $C_{1,2}=C_{2,3}=0$ and thus $C_{k,k+1}=0$ $\forall k$. We have shown that $C_{k,n+k}=0$ $\forall n\geqq 1$ and $\forall k$. But $C_{i,j}=C_{j,i}^*$, so $C_{i,j}=0$ $\forall i\neq j$.

Because $E_{k+1,k}=F_{k+1,k}$, we find that $C_{k,k}=C_{0,0}$ $\forall k$. Finally, since $E_{2,0}=F_{2,0}$, we have $AC_{0,0}=C_{2,2}A=C_{0,0}A$, i.e., $C_{0,0}\in\mathfrak{A}'$. Therefore $\mathcal{R}(B)'=\{(\delta_{i,j}D)_{i,j=-\infty}^{\infty}:D\in\mathfrak{A}'\}$. Since $\mathfrak{A}$ is *-isomorphic to $M_\infty(\mathfrak{A})$, the proof is complete.

REMARK 3. Russo has shown (cf. [4, Theorem 1]) that if A is a unimodular contraction and $\mathcal{R}(A)$ is finite, then A is unitary. It follows if A is any unimodular contraction, then $\mathcal{R}(A)$ is of the form $\mathfrak{A}\oplus\mathfrak{B}$, where $\mathfrak{A}$ is abelian and $\mathfrak{B}$ is properly infinite.

References

1. H. Behncke, *Generators of W^*-algebras* (to appear).

2. J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien* (*Algèbres de von Neumann*), Cahiers Scientifiques, fasc. 25, Gauthier-Villars, Paris, 1957. MR **20** #1234.

3. C. M. Pearcy, *On certain von Neumann algebras which are generated by partial isometries*, Proc. Amer. Math. Soc. **15** (1964), 393–395. MR **28** #4380.

4. B. Russo, *Unimodular contractions in Hilbert space*, Pacific J. Math. **26** (1968), 163–169. MR **37** #6769.

5. D. Topping, *Transcendental quasinilpotents in operator algebras*, J. Functional Anal. **2** (1968), 342–351. MR **38** #1535.

6. ———, *Lectures on von Neumann algebras*, Lecture notes, Tulane University, 1969.

7. W. Wogen, *On generators for von Neumann algebras*, Bull. Amer. Math. Soc. **75** (1969), 95–99. MR **38** #5020.

University of North Carolina, Chapel Hill, North Carolina 27514