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INVARIANT MEASURES ON LOCALLY COMPACT SEMIGROUPS

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ABSTRACT. The main result of this paper shows that a locally compact abelian semigroup is embeddable as an open subsemigroup of a locally compact abelian group G if and only if the translations $x \mapsto x+y$ are open maps and there exists a nonnegative regular measure μ on S such that $\mu(U)=\mu(x+U)>0$ for every open set U and x in S.

Our main result is a somewhat stronger statement than the above in that we show that whenever such a measure exists it is the restriction to S of the Haar measure on G. This provides a partial answer to a question raised by J. H. Williamson in [**5**, §5]. We follow the terminology of [**5**] as regards semigroups and the measure theoretic terminology of [**3**]. In particular:

A locally compact abelian semigroup S is an abelian semigroup (not necessarily having a unit) which is a locally compact Hausdorff space such that for each y in S the map $x \mapsto x+y$ is continuous. We say that a locally compact abelian semigroup S is embeddable in a locally compact group G if there exists a bicontinuous semigroup monomorphism ϕ mapping S into G. The following proposition is of independent interest (see [**4**, Theorem 2.1 and Lemma 1.3]).

PROPOSITION. *Let S be a locally compact abelian semigroup. The following conditions on S are equivalent.*

(1) *S is a cancellation semigroup and for each open subset U of S, $x+U$ is open for each x in S.*

(2) *S is embeddable as an open subsemigroup of a locally compact group G.*

PROOF. It is clear that (2) implies (1). To show that (1) implies (2) let R be the equivalence relation on $S \times S$ defined by $(x, y)R(x_0, y_0)$ if and only if $x+y_0=y+x_0$. It is well known and easy to show that $G=S\times S/R$ is an abelian group. For x in S let $\phi(x)$ be the equivalence class $\{(x+y, y): y \in S\}$. The map $\phi: x \mapsto \phi(x)$ is one-one and satisfies $\phi(x+y)=\phi(x)+\phi(y)$. We now define a topology on G. For x in S let $\mathcal{B}(x)$ be the neighbourhood filter of x. Choose some x_0 in S and for

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each x in G let $x \cdot \mathcal{B}$ be the filter on G generated by the filter base $\{\phi(U)-\phi(x_0)+x: U\in\mathcal{B}(x_0)\}$. We first show that if $x\in S$, then $\phi(x)\cdot\mathcal{B}$ is generated by the filter base $\{\phi(U): U\in\mathcal{B}(x)\}$. Given $U\in\mathcal{B}(x_0)$, $U+x\in\mathcal{B}(x_0+x)$ and the continuity of $z\mapsto z+x_0$ at x means there is a $V\in\mathcal{B}(x)$ with $V+x_0\subset U+x$ and so $\phi(V)\subset\phi(U)-\phi(x_0)+\phi(x)$. Conversely given $V\in\mathcal{B}(x)$, $V+x_0\in\mathcal{B}(x+x_0)$ and the continuity of $z\mapsto z+x$ at x_0 means there is a $U\in\mathcal{B}(x_0)$ such that $U+x\subset V+x_0$ and so $\phi(U)-\phi(x_0)+\phi(x)\subset\phi(V)$. Therefore $\phi(x)\cdot\mathcal{B}$ is generated by $\{\phi(U): U\in\mathcal{B}(x)\}$. We now show that there is a unique topology on G such that for each x in G, $x\cdot\mathcal{B}$ is the neighbourhood filter of x. For this it is sufficient by [1, Chapitre 1, §1, No. 1] to show for each x in G:

(i) if $V\in x\cdot\mathcal{B}$ then $x\in V$;

(ii) if $V\in x\cdot\mathcal{B}$ then there is a $W\in x\cdot\mathcal{B}$ such that $y\in W$ implies $V\in y\cdot\mathcal{B}$.

Clearly (i) is satisfied, so we show (ii). Let $V\in x\cdot\mathcal{B}$. Then there is an open neighbourhood U of x_0 such that $W=\phi(U)-\phi(x_0)+x\subset V$. If $y\in W$, there is a $u\in U$ with $y=\phi(u)-\phi(x_0)+x$ and there is a $V'\in\mathcal{B}(x_0)$ such that $V'+u\subset U+x_0$. Thus

$$\begin{aligned}\phi(V') - \phi(x_0) + y &= \phi(V') - \phi(x_0) + \phi(u) - \phi(x_0) + x \\ &\subset \phi(U) - \phi(x_0) + x = W\end{aligned}$$

and therefore $W\in y\cdot\mathcal{B}$ so that (ii) is satisfied.

It is clear that G with this topology is a Hausdorff space and that the maps $x\mapsto x+y$ are continuous. Moreover for each x in S, $\phi(x)\cdot\mathcal{B}$ is generated by $\{\phi(U): U\in\mathcal{B}(x)\}$ so that ϕ is a topological embedding and $\phi(S)$ is an open subset of G. The continuity and openness of the map $x\mapsto x+y$ for each y in G together with the local compactness of S imply that G is locally compact. Thus G is a locally compact semigroup which is a group. A theorem of R. Ellis [2, Theorem 2] shows that G is a locally compact group. This completes the proof.

THEOREM. *Let S be a locally compact abelian semigroup and μ a nonnegative regular measure on S. Suppose that S and μ satisfy the following condition.*

(*) *For each open set U, $x+U$ is open for each x in S and $\mu(x+U)=\mu(U)>0$.*

Then S is embeddable as an open subsemigroup in a locally compact abelian group G and μ is the restriction of the Haar measure of G to S. Conversely if S is an open subsemigroup of a locally compact abelian group G, and if μ is the restriction to S of the Haar measure of G, then S is a locally compact abelian semigroup and S and μ satisfy condition (*).

Proof. First suppose S and μ are given and satisfy (*). We begin by showing that S is a cancellation semigroup. If not there are x, y z in S such that $y+x=y+z$ and $x\neq z$. There are open relatively compact neighbourhoods U of x and V of z such that $U\cap V$ is empty. Now

$$\mu((y+U)\cup(y+V)) = \mu(y+(U\cup V)) = \mu(U\cup V)$$
$$= \mu(U)+\mu(V) = \mu(y+U)+\mu(y+V).$$

Since regular measures are by definition finite on compacta it follows that $\mu((y+U)\cap(y+V))=0$ which is a contradiction because $(y+U)\cap(y+V)$ is a neighbourhood of $y+x$. Thus S satisfies the hypotheses of the above proposition so S is embeddable as an open subsemigroup of a locally compact abelian group G. In the following we identify S with its image in G.

Let $\mathcal{K}_S(G)$ be the continuous complex-valued functions on G which are zero outside of S and have compact support. Observe that the invariance property of μ means that if $f\in\mathcal{K}_S(G)$ then for each y in S,

$$\int_S f(x-y)d\mu(x) = \int_S f\,d\mu.$$

Choose $g\in\mathcal{K}_S(G)$ with $g\geqq 0$ and $\int g\,d\lambda=1$ where λ is the Haar measure on G. Then using the Fubini Theorem [3, p. 153] we have

$$\begin{aligned}\int_S f\,d\mu &= \int_S f(x-y)\,d\mu(x)\int_G g(y)\,d\lambda(y)\\ &= \int_S\int_G f(x-y)g(y)\,d\lambda(y)\,d\mu(x)\\ &= \int_S\int_S f(y)g(x-y)\,d\lambda(y)\,d\mu(x)\\ &= \int_S g(x-y)\,d\mu(x)\int_S f(y)\,d\lambda(y)\\ &= c\int_S f\,d\lambda\end{aligned}$$

where $c=\int_S g\,d\mu$. It follows now that for any Borel subset $E\subset S$, $\mu(E)=c\lambda(E)$ [3, p. 129]. This completes the proof of the first statement.

If S is an open subsemigroup of a locally compact abelian group, then it is clear that S is a locally compact abelian semigroup. Moreover since S is open the restriction of the Haar measure on G to S yields a nonnegative regular measure μ such that condition (*) is satisfied.

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