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s ADMITS AN INJECTIVE METRIC

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ABSTRACT. There is an injective metric space homeomorphic with a countably infinite product of lines.

Introduction. The category of metric spaces and contractions (mappings which increase no distance) has injective objects, indeed injective envelopes [3], but little is known about their structure. Their geometry is in a sense the worst possible; for E to be injective requires that however one attaches a plate to E consistent with the triangle inequality, it can be contracted into E. Their topology is in a sense the best possible. They are topologically injective and topologically complete [2]; the locally compact ones are locally triangulable at every homotopically stable point [3]. Beyond dimension 2 [3], it is quite unknown which finite polyhedra admit injective metrics.

This note adds one example: an injective metric space D is homeomorphic with the Banach space c of convergent sequences, and therefore (by Kadec [4] and Anderson [1]) with a product of lines s, and with many other geometrically different spaces. Such an example cannot be isometric with a Banach space [2]. The homeomorphism constructed from c to D is not uniformly continuous, and it bends all straight lines in c except for one parallel pencil. How much of that is necessary is quite unknown.

Proof. The example is the space D of sequences (x_i) of real numbers converging to a limit λ from above, i.e. $\lambda = \inf x_i$, with the distance $\sup|x_i - y_i|$ induced on D as a subspace of l_∞. l_∞ is injective [2].

LEMMA. *The pointwise maximum of two contractions from a metric space to the real line is a contraction.*

PROOF. For two contractions f, g, and two points x, y, one of the four numbers $f(x)$, $f(y)$, $g(x)$, $g(y)$ is largest, say $f(x)$. Then $f(x) - d(x,y) \leqq f(y) \leqq (f \vee g)(y) \leqq f(x) = (f \vee g)(x)$; so $|(f\vee g)(x) - (f \vee g)(y)| \leqq d(x,y)$.

THEOREM 1. *D is a retract of l_∞ and therefore injective.*

PROOF. For $x = (x_i)$ in l_∞, let $\lambda(x) = \limsup x_i$ and $p_i(x) = x_i \vee \lambda(x)$.

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By the lemma, p_i is a contraction. Hence $P(x)=(p_i(x))$ defines a contraction, evidently a retraction upon D.

THEOREM 2. *D is homeomorphic with c.*

PROOF. Since c is evidently homeomorphic with $c_0\times R$ (c_0 the space of sequences converging to 0) and D with $D_0\times R$ (D_0 the nonnegative sequences converging to 0), it suffices to give a homeomorphism $f:c_0\to D_0$. For $(x_1, x_2, \cdots)\in c_0$, put $x_0=0$. Then $f(x)=y$ is defined by $y_i=x_{i-1}^-+\sum_{j=i}^{\infty} 2^{i-j}x_j^+$ (where $x^-=-x\vee 0$, $x^+=x\vee 0$). So f is Lipschitzian, increasing no distance by more than a factor of 3. By inspection, f takes values in D_0.

To describe the inverse of f, note first that for any sequence $\{z_i: i=1, 2, \cdots\}$, the equations $x_0=0$, and $2x_{i-1}^-+2x_i^+-x_i^-=z_i$ determine a unique sequence $\{x_i: i=0, 1, 2, \cdots\}$. If $y\in D_0$, define x by $x_0=0$ and

$$(*)\qquad 2x_{i-1}^- + 2x_i^+ - x_i^- = 2y_i - y_{i+1}$$

and put $g(y)=(x_1, x_2, \cdots)$.

We shall now prove the inequalities $x_i^+\leqq y_i$ and $x_i^-\leqq y_{i+1}$ for $y\in D_0$ and $i>0$. The former is immediate since either $x_i^+=0$ or from (*) $2x_i^+\leqq 2y_i$. To prove the latter, assume to begin with that $x_{i-1}^-=0$. Then $x_i^-=0$ or $x_i^-=y_{i+1}-2y_i\leqq y_{i+1}$. Suppose now that $x_{i-1}^-, x_{i-2}, \cdots, x_{i-k}^->0$, but $x_{i-k-1}^-=0$. Then $x_{i-1}^+=x_{i-2}^+=\cdots=x_{i-k}^+=0$, and a linear combination of equations (*) for $i, i-1, \cdots, i-k$ gives

$$2x_i^+ - x_i^- = 2^{k+1}y_{i-k} - y_{i+1}.$$

Again either $x_i^-=0$ or $x_i^-\leqq y_{i+1}$.

The inequalities just proved show that $x_i\to 0$; i.e. $g(y)\in c_0$. To prove that g is continuous at y, let $\epsilon>0$ be given. Choose n so that $i\geqq n\Rightarrow y_i<\epsilon/3$. If $z\in D_0$ and $\|z-y\|<\delta=\epsilon/3n$, then (*) shows by induction that $|g(z)_i-g(y)_i|=|g(z)_i-x_i|<3i\delta\leqq\epsilon$ for $i=0, 1, \cdots, n$. For $i>n$ we have

$$|g(z)_i| \leqq \max\{z_i, z_{i+1}\} < \max\{y_i, y_{i+1}\} + \delta \leqq \epsilon/3 + \delta \leqq 2\epsilon/3$$

and

$$|g(y)_i| \leqq \max\{y_i, y_{i+1}\} \leqq \epsilon/3.$$

Thus $\|g(z)-g(y)\|<\epsilon$.

Finally we must show that f and g are inverses. If $y=f(x)$ we see from the definition of f that the components of x satisfy (*). Hence $x=g(y)$; i.e., gf is the identity on c_0. If $x=g(y)$ and $\bar{y}=f(x)$, then (*)

and the definition of f lead to $2\bar{y}_i-\bar{y}_{i+1}=2y_i-y_{i+1}$. Hence $\bar{y}_{i+1}-y_{i+1}=2(\bar{y}_i-y_i)=2^i(\bar{y}_1-y_1)$. Since both $\bar{y}$ and y are in D_0, this implies $\bar{y}=y$. Thus fg is the identity on D_0. This completes the proof.

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