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ABSTRACT MARTINGALES IN BANACH SPACES

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ABSTRACT. The concept of martingale is generalized from probability theory to the setting of Banach spaces. Convergent martingales are characterized. An application to a Radon-Nikodym theorem for vector measures is given.

1. **Abstract martingales.** Let X be a Banach space and $\{E_\tau, \tau \in I\}$ be a uniformly bounded net of continuous linear projections of X into itself satisfying $E_\tau E_{\tau_1} = E_{\tau_1} E_\tau = E_{\tau_1}$ for $\tau \geqq \tau_1 \in I$. A net $\{x_\tau, \tau \in I\} \subset X$ indexed by the same directed set I will be called an abstract martingale and denoted by $\{x_\tau, E_\tau, \tau \in I\}$ if $E_{\tau_1}(x_{\tau_2}) = X_{\tau_1}$ for $\tau_1, \tau_2 \in I$, $\tau_1 \leqq \tau_2$. Clearly abstract martingales are generalizations of the martingales of probability theory [**2**], [**5**], and [**8**]. On the other hand there are many examples of abstract martingales which do not arise as martingales in the sense of probability theory (see [**3**, pp. 426–427]). The purpose of this note is to characterize strongly convergent abstract martingales and to indicate briefly some applications including a new Radon-Nikodym theorem for vector valued measures.

THEOREM 1. *Let $\{x_\tau, E_\tau, \tau \in I\}$ be an abstract martingale in a Banach space X. Then $\{x_\tau, E_\tau, \tau \in I\}$ is strongly convergent (i.e. $\lim_\tau x_\tau$ exists strongly in X) if and only if there exists a weakly compact set $K \subset X$ such that for each $\epsilon > 0$ there exists a $\tau_\epsilon \in I$ such that $\tau \in I$, $\tau \geqq \tau_\epsilon$ implies $x_\tau \in K + \epsilon U$ $(= \{k + \epsilon u : k \in K, u \in U\})$ where U is the open unit ball of X.*

PROOF. The necessity is immediate: let $K = \{\lim_\tau x_\tau\}$. Then $\{x_\tau, \tau \in I\}$ is eventually in $K + \epsilon U$ for every choice of ϵ. To prove the sufficiency of the condition, let K be as in the hypothesis and select $\{\tau_n\} \subset I$ by choosing τ_1 such that $\tau \geqq \tau_1$ implies $x_\tau \in K + U$ and $\tau_n \geqq \tau_{n-1}$ such that $x_\tau \in K + (1/n)U$ for $\tau \geqq \tau_n$. Now for each $\tau \in I$, choose z_τ according to the following criteria:

(i) if $\tau \geqq \tau_n$ for all n, then $x_\tau \in K$ and z_τ is taken to be x_τ;

(ii) if $\tau \geqq \tau_{n_0}$ and it is not the case that $\tau \geqq \tau_{n_0+1}$, choose $z_\tau \in K$ such that $\|z_\tau - x_\tau\| < 1/n_0$;

(iii) if there exists no n such that $\tau \geqq \tau_n$, choose $z_\tau \in K$ arbitrarily.

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Now consider the net $\{z_\tau, \tau\in I\}\subset K$. Since K is weakly compact, there exists a subnet $\{z_\alpha, \alpha\in A\}$ of $\{z_\tau, \tau\in I\}$ converging weakly to some point $x\in K$. Now let $f: A\to I$ be a function which guarantees that $\{z_\alpha, \alpha\in A\}$ is a subnet of $\{z_\tau, \tau\in I\}$ [6, p. 70] and define $\{x_\alpha, \alpha\in A\}$ by $x_\alpha = x_{f(\alpha)}$. Then $\{x_\alpha, \alpha\in A\}$ is a subnet of $\{x_\tau, \tau\in I\}$ and $\|x_\alpha - z_\alpha\| = \|x_{f(\alpha)} - z_{f(\alpha)}\|$. Moreover if $x^*\in X^*$, the space of bounded linear functionals on X, one has

$$\lim_\alpha |x^*(x_\alpha - x)| \leqq \lim_\alpha |x^*(x_\alpha - z_\alpha)| + \lim_\alpha |x^*(z_\alpha - x)|$$

$$\leqq \|x^*\| \lim_n \frac{1}{n} + 0 = 0.$$

Hence $\lim_\alpha x_\alpha = x$ weakly in X. Also since $\{x_\alpha, \alpha\in A\}$ is a subnet of $\{x_\tau, \tau\in I\}$, $x_\tau = \lim_\alpha E_\tau(x_\alpha)$ strongly for all $\tau\in I$. Accordingly if $\tau\in I$ and $x^*\in X^*$,

$$x^*(x_\tau - E_\tau(x)) = x^*(E_\tau(x_\tau - x)) = \lim_\alpha x^*E_\tau(x_\alpha - x_\alpha) = 0,$$

since $\lim_\alpha x_\alpha = x$ weakly and $x_\tau = \lim_\alpha E_\tau(x_\alpha)$ strongly. Hence $x_\tau = E_\tau(x)$ for all $\tau\in I$.

Finally it will be shown that $\lim_\tau x_\tau = x$ strongly in X. Let $M = \{z\in X: E_\tau(z) = z$ for some $\tau\in I\}$. The facts that I is directed and that $E_\tau E_{\tau_1} = E_{\tau_1}E_\tau = E_{\tau_1}$ for $\tau\geqq\tau_1$ ensure that M is a linear manifold in X. But, since $\lim_\alpha x_\alpha = x$ weakly and $\{x_\alpha\}\subset M$, $x\in$weak closure of M and therefore to the strong closure of the linear manifold M. Now let $P = \sup_\tau \|E_\tau\|$ and $\epsilon > 0$ be given. Choose $y\in M$ such that $\|x - y\| < \epsilon/P+1$. Selecting $\tau_0\in T$ such that $E_{\tau_0}(y) = y$, one finds that for $\tau\geqq\tau_0$, $E_\tau(y) = y$ since $E_\tau(y) = E_\tau E_{\tau_0}(y) = E_{\tau_0}(y) = y$. Hence for $\tau\geqq\tau_0$,

$$\|x_\tau - x\| = \|E_\tau(x) - x\| \leqq \|E_\tau(x) - y\| + \|y - x\|$$

$$= \|E_\tau(x - y)\| + \|y - x\| < P\epsilon/(P+1) + \epsilon/(P+1) = \epsilon.$$

Q.E.D.

A considerable shortening of the proof of Theorem 1 results in

COROLLARY 2. *An abstract martingale is strongly convergent if and only if it is weakly convergent.*

Also immediate is

COROLLARY 3. *An abstract martingale in a reflexive Banach space is convergent if and only if it is bounded.*

2. **Applications to martingales and integral representation of vector measures.** If X is a reflexive Banach space, and (Ω, Σ, μ) is a finite measure space, Scalora and Chatterji have shown that a martingale $\{f_n, B_n\}$ in $L^p(\Omega, \Sigma, \mu, X)$ $(=L^p(X))$ converges for $1<p<\infty$ if and only if $\{f_n, B_n\}$ is bounded [**2**, Theorem 3]. Since the spaces $L^p(X)$ $(1<p<\infty)$ are reflexive, for reflexive X, Corollary 3 contains this result as a special case. In the case $p=1$, Chatterji and Scalora prove that a martingale $\{f_n, B_n\}$ in $L^1(X)$ is convergent if it is bounded and uniformly integrable for reflexive Banach spaces X. But, as Chatterji points out [**2**, p. 145], this assumption guarantees that $\{f_n, B_n\}$ lies in a weakly compact subset of $L^1(X)$. Thus Theorem 1 and its corollary contain the full Chatterji-Scalora theorem on mean convergence of martingales in $L^p(X)$ $(1\leqq p<\infty)$. Of course this theorem gives no direct information on almost sure convergence of martingales. On the other hand such information is not to be expected from a theorem of the nature of Theorem 1.

The connection between martingales and derivatives of set functions is well known [**8**]. The final considerations of this note are devoted to that subject.

Let (Ω, Σ, μ) be a finite measure space. A partition $\pi=\{E_n\}$ is a finite disjoint collection of sets in Σ such that $\bigcup_n E_n=\Omega$. The collection of partitions P becomes a directed set if one defines $\pi_1\leqq\pi_2$ if $E\in\pi_1$ implies E is a union of members of π_2. Now let F be a μ-continuous countably additive set function defined on Σ with values in a Banach space X. Define for each partition $\pi=\{E_n\}$ the simple function

$$F_\pi = \sum_\pi \frac{F(E_n)}{\mu(E_n)}\chi_{E_n}, \qquad (0/0) = 0,$$

where χ_{E_n} is the indicator function of $E_n\in\Sigma$. Then, as Rønnow [**7**] has shown for the case $p=1$ (the same argument holds for all $p\geqq 1$) there exists $f\in L^p(\Omega, \Sigma, \mu, X)$ $(1\leqq p<\infty)$ such that

$$F(E) = \int_E f d\mu, \qquad E \in \Sigma, \text{ (Bochner)}$$

if and only if the net $\{F_\pi, \pi\in p\}$ is a Cauchy net in $L^p(\Omega, \Sigma, \mu, X)$. Now the projections E_π defined on $L^p(\Omega, \Sigma, \mu, X)$ for each partition $\pi=\{E_n\}$ by

$$E_\pi(f) = \sum_\pi \frac{\int_{E_n} f d\mu}{\mu(E_n)}\chi_{E_n}$$

for $f \in L^p(\Omega, \Sigma, \mu, X)$ are contractions satisfying $E_\pi E_{\pi_1} = E_{\pi_1} E_\pi = E_{\pi_1}$ if $\pi \geqq \pi_1$. Now, evidently if F is as above, then $\{F_\pi, E_\pi, \pi \in P\}$ is an abstract martingale in $L^p(\Omega, \Sigma, \mu, X)$, combining these facts with Theorem 1 results in the following general Radon-Nikodym theorem.

THEOREM 4. *Let (Ω, Σ, μ) be a finite measure space and X be a Banach space. Let F be a μ-continuous countably additive X valued set function defined on Σ. Then there exists $f \in L^p(\Omega, \Sigma, \mu, X)$ $(1 \leqq p < \infty)$ such that*

$$F(E) = \int_E f d\mu, \qquad E \in \Sigma,$$

if and only if there exists a weakly compact set $K \subset L^p(\Omega, \Sigma, \mu, X)$ with the property that for each $\epsilon < 0$ there exists a partition π_0 such that $\pi \geqq \pi_0$ implies $F_\pi \in K + \epsilon U$ where U is the open unit ball of $L^p(\Omega, \Sigma, \mu, X)$.

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