

A TOPOLOGICAL CHARACTERIZATION OF THE DILATION IN E^n

L. S. HUSCH¹

ABSTRACT. A topological characterization is given to determine whether a homeomorphism of Euclidean n -space, $n \neq 4, 5$, is topologically equivalent to the dilation $x \rightarrow \frac{1}{2}x$.

B. v. Kerékjártó [6] and T. Homma and S. Kinoshita [4] have given topological characterizations of the dilations in 2-dimensional and 3-dimensional Euclidean space, E^2 and E^3 , respectively.

THEOREM. *Let h be an orientation-preserving homeomorphism of E^n onto itself, $n \neq 4, 5$; let h' be the unique extension to the n -sphere, $S^n = E^n \cup \{\infty\}$, and let d be a metric for S^n . The following are equivalent:*

- (1) *h is topologically equivalent to the dilation $x \rightarrow \frac{1}{2}x$.*
- (2) *h' is regular or has equicontinuous powers for each $x \in S^n - \{0, \infty\}$ but not at 0 or ∞ ;—i.e. for each $\epsilon > 0$ there exists $\delta > 0$ such that whenever $d(x, y) < \delta$, $d(h^m(x), h^m(y)) < \epsilon$ for every integer m .*
- (3) *For all x , $\lim_{i \rightarrow +\infty} h^i(x) = 0$ and for all $x \neq 0$, $\lim_{i \rightarrow -\infty} h^i(x) = \infty$.*
- (4) *For each compact subset $C \subset E^n$, $\lim_{i \rightarrow +\infty} h^i(C) = 0$ and for each compact subset C in $E^n - \{0\}$, $\lim_{i \rightarrow -\infty} h^i(C) = \infty$.*

It is clear that (1) implies each of the remaining three conditions. The last three conditions are known to be equivalent [7, p. 223]. Suppose $n \geq 6$. Let G be the group of automorphisms of E^n generated by h . Note that (2) implies that $h(0) = 0$. Hence we can regard G as a group of automorphisms of $U = E^n - \{0\}$. Let $p: U \rightarrow U/G$ be the natural projection onto the orbit space. It follows from (4) and [8], that p is a covering projection and G is the group of covering transformations. Hence U/G is a closed connected n -manifold.

We wish to show that U/G is homeomorphic to $S^1 \times S^{n-1}$. By [9], U/G has the homotopy type of a finite complex. Hence by [3, p. 298], the homotopy classes of maps of U/G into S^1 are in a 1-1 correspondence with $H^1(U/G) = \text{integers}$. In particular, there exists $f: U/G \rightarrow S^1$ which is essential. Suppose that the induced map $f_*: \pi_1(U/G) \rightarrow \pi_1(S^1)$

Presented to the Society, November 20, 1970; received by the editors July 17, 1970.

AMS 1970 subject classifications. Primary 57E20, 57E30; Secondary 57A15.

Key words and phrases. Dilation, homeomorphisms of E^n , regular homeomorphisms, equicontinuous powers, action of integers on E^n .

¹ Research supported in part by N.S.F. Grant GP-15357.

is not onto. Let $g: S^1 \rightarrow S^1$ be the covering space corresponding to $f_*(\pi_1(U/G))$. By [3, p. 257], there exists $f': U/G \rightarrow S^1$ such that $gf' = f$ and hence $f'_*: \pi_1(U/G) \rightarrow \pi_1(S^1)$ is onto and hence an isomorphism. Therefore, let us assume that f_* is an isomorphism. Note that the infinite cyclic covering of U/G corresponding to f_* is U , which has the homotopy type of S^{n-1} . Recall that the Whitehead group of the integers is 0 [2]. Hence f can be homotoped to a fibration by [10, p. 11] (although Theorem 4.1 of [10] is stated in the differential category, it is also valid in the topological category; see [10, p. 2]) with fibre which is a manifold of the homotopy type of S^{n-1} and hence is S^{n-1} [1]. Since U/G is orientable, U/G is homeomorphic to $S^1 \times S^{n-1}$ since an orientation-preserving homeomorphism of S^{n-1} is isotopic to the identity [5].

Consider $p^{-1}(\{x\} \times S^{n-1})$ for some $x \in S^1$. Since $p: p^{-1}(\{x\} \times S^{n-1}) \rightarrow \{x\} \times S^{n-1}$ is a covering projection, it follows that $p^{-1}(\{x\} \times S^{n-1})$ is a countable collection of disjoint $(n-1)$ -spheres S_σ such that $p|_{S_\sigma}$ is a homeomorphism for each S_σ . Note that $h(S_\sigma) \cap S_\sigma = \emptyset$. We proceed now as in [4] to complete the proof; we include the proof for completeness.

Let S_t denote the sphere in E^n with center at 0 and radius t . Let $\beta: S^{n-1} \times (0, \infty) \rightarrow E^n - \{0\}$ be a homeomorphism such that $\beta(S^{n-1} \times \{t\}) = S_t$ and $\beta(\{x\} \times (0, \infty))$ is a straight line in E^n .

There is a homeomorphism γ of E^n such that $\gamma(0) = 0$, $\gamma(S_\sigma) = S_2$, and $\gamma(h(S_\sigma)) = S_1$. Define $\delta: S^{n-1} \rightarrow S^{n-1}$ by $\beta^{-1}\gamma h\gamma^{-1}\beta(x, 2) = (\delta(x), 1)$. Since δ is orientation-preserving, it follows from [5] that there is an isotopy $\delta_t: S^{n-1} \rightarrow S^{n-1}$, $0 \leq t \leq 1$, such that $\delta_0 = \delta$ and $\delta_1 = \text{identity}$.

Define $F_0: \beta(S^{n-1} \times [1, 2]) \rightarrow E^n$ by $F_0(\beta(x, 1+t)) = \beta(\delta_t(x), 1+t)$. Extend F_0 to F , a homeomorphism of E^n , by $F(0) = 0$ and $F(\beta(x, r)) = \gamma^{-1}h^q\gamma F_0(\beta(x, 2^q r))$ where q is the unique integer such that $1 < 2^q r \leq 2$. One easily sees that $F^{-1}\gamma h\gamma^{-1}F(\beta(x, r)) = \beta(x, \frac{1}{2}r)$.

REFERENCES

1. E. H. Connell, *A topological H-cobordism theorem for $n \geq 5$* , Illinois J. Math. 11 (1967), 300-309. MR 35 #3673.
2. G. Higman, *The units of group rings*, Proc. London Math. Soc. (2) 46 (1940), 231-248. MR 2, 5.
3. P. J. Hilton and S. Wylie, *Homology theory: An introduction to algebraic topology*, Cambridge Univ. Press, New York, 1960. MR 22 #5963.
4. T. Homma and S. Kinoshita, *On a topological characterization of the dilatation in E^3* , Osaka Math. J. 6 (1954), 135-144. MR 16, 160.
5. W. C. Hsiang and J. L. Shaneson, *Fake tori, the annulus conjecture and the conjectures of Kirby*, Proc. Nat. Acad. Sci. U.S.A. 62 (1969), 687-691.
6. B. v. Kerékjártó, *Topologische Charakterisierung der linearen Abbildungen*, Acta Litt. Acad. Sci. Szeged. 6 (1934), 235-262.

7. S. Kinoshita, *On quasi-translations in 3-space*, *Topology of 3-Manifolds and Related Topics* (Proc. Univ. of Georgia Inst., 1961), Prentice-Hall, Englewood, Cliffs, N. J., 1962, pp. 223–226. MR 25 #3116.

8. ———, *Notes on covering transformation groups*, *Proc. Amer. Math. Soc.* 19 (1968), 421–424. MR 36 #5921.

9. R. C. Kirby and L. C. Siebenmann, *On the triangulation of manifolds and the Hauptvermutung*, *Bull. Amer. Math. Soc.* 75 (1969), 742–749. MR 39 #3500.

10. L. C. Siebenmann, *A total Whitehead torsion obstruction to fibering over the circle*, *Comment. Math. Helv.* 45 (1970), 1–48.

VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY, BLACKSBURG,
VIRGINIA 24061