PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, No. 1, April 1971

A TOPOLOGICAL CHARACTERIZATION OF THE DILATION IN E^n

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ABSTRACT. A topological characterization is given to determine whether a homeomorphism of Euclidean n-space, $n \neq 4, 5$, is topologically equivalent to the dilation $x \to \frac{1}{2}x$.

B. v. Kerékjártó [6] and T. Homma and S. Kinoshita [4] have given topological characterizations of the dilations in 2-dimensional and 3-dimensional Euclidean space, E^2 and E^3, respectively.

THEOREM. *Let h be an orientation-preserving homeomorphism of E^n onto itself, $n \neq 4, 5$; let h' be the unique extension to the n-sphere, $S^n = E^n \cup \{\infty\}$, and let d be a metric for S^n. The following are equivalent:*

(1) *h is topologically equivalent to the dilation $x \to \frac{1}{2}x$.*

(2) *h' is regular or has equicontinuous powers for each $x \in S^n - \{0, \infty\}$ but not at 0 or ∞;—i.e. for each $\epsilon > 0$ there exists $\delta > 0$ such that whenever $d(x, y) < \delta$, $d(h'^m(x), h'^m(y)) < \epsilon$ for every integer m.*

(3) *For all x, $\lim_{i \to +\infty} h^i(x) = 0$ and for all $x \neq 0$, $\lim_{i \to -\infty} h^i(x) = \infty$.*

(4) *For each compact subset $C \subset E^n$, $\lim_{i \to +\infty} h^i(C) = 0$ and for each compact subset C in $E^n - \{0\}$, $\lim_{i \to -\infty} h^i(C) = \infty$.*

It is clear that (1) implies each of the remaining three conditions. The last three conditions are known to be equivalent [7, p. 223]. Suppose $n \geqq 6$. Let G be the group of automorphisms of E^n generated by h. Note that (2) implies that $h(0) = 0$. Hence we can regard G as a group of automorphisms of $U = E^n - \{0\}$. Let $p: U \to U/G$ be the natural projection onto the orbit space. It follows from (4) and [8], that p is a covering projection and G is the group of covering transformations. Hence U/G is a closed connected n-manifold.

We wish to show that U/G is homeomorphic to $S^1 \times S^{n-1}$. By [9], U/G has the homotopy type of a finite complex. Hence by [3, p. 298], the homotopy classes of maps of U/G into S^1 are in a 1-1 correspondence with $H^1(U/G) =$ integers. In particular, there exists $f: U/G \to S^1$ which is essential. Suppose that the induced map $f_*: \pi_1(U/G) \to \pi_1(S^1)$

Presented to the Society, November 20, 1970; received by the editors July 17, 1970.

AMS 1970 *subject classifications*. Primary 57E20, 57E30; Secondary 57A15.

Key words and phrases. Dilation, homeomorphisms of E^n, regular homeomorphisms, equicontinuous powers, action of integers on E^n.

[1] Research supported in part by N.S.F. Grant GP-15357.

is not onto. Let $g: S^1 \to S^1$ be the covering space corresponding to $f_*(\pi_1(U/G))$. By [3, p. 257], there exists $f': U/G \to S^1$ such that $gf' = f$ and hence $f'_*: \pi_1(U/G) \to \pi_1(S^1)$ is onto and hence an isomorphism. Therefore, let us assume that f_* is an isomorphism. Note that the infinite cyclic covering of U/G corresponding to f_* is U, which has the homotopy type of S^{n-1}. Recall that the Whitehead group of the integers is 0 [2]. Hence f can be homotoped to a fibration by [10, p. 11] (although Theorem 4.1 of [10] is stated in the differential category, it is also valid in the topological category; see [10, p. 2]) with fibre which is a manifold of the homotopy type of S^{n-1} and hence is S^{n-1} [1]. Since U/G is orientable, U/G is homeomorphic to $S^1 \times S^{n-1}$ since an orientation-preserving homeomorphism of S^{n-1} is isotopic to the identity [5].

Consider $p^{-1}(\{x\} \times S^{n-1})$ for some $x \in S^1$. Since $p: p^{-1}(\{x\} \times S^{n-1}) \to \{x\} \times S^{n-1}$ is a covering projection, it follows that $p^{-1}(\{x\} \times S^{n-1})$ is a countable collection of disjoint $(n-1)$-spheres S_σ such that $p | S_\sigma$ is a homeomorphism for each S_σ. Note that $h(S_\sigma) \cap S_\sigma = \emptyset$. We proceed now as in [4] to complete the proof; we include the proof for completeness.

Let S_t denote the sphere in E^n with center at 0 and radius t. Let $\beta: S^{n-1} \times (0, \infty) \to E^n - \{0\}$ be a homeomorphism such that $\beta(S^{n-1} \times \{t\}) = S_t$ and $\beta(\{x\} \times (0, \infty))$ is a straight line in E^n.

There is a homeomorphism γ of E^n such that $\gamma(0) = 0$, $\gamma(S_\sigma) = S_2$, and $\gamma(h(S_\sigma)) = S_1$. Define $\delta: S^{n-1} \to S^{n-1}$ by $\beta^{-1}\gamma h \gamma^{-1}\beta(x, 2) = (\delta(x), 1)$. Since δ is orientation-preserving, it follows from [5] that there is an isotopy $\delta_t: S^{n-1} \to S^{n-1}$, $0 \leqq t \leqq 1$, such that $\delta_0 = \delta$ and $\delta_1 = $ identity.

Define $F_0: \beta(S^{n-1} \times [1, 2]) \to E^n$ by $F_0(\beta(x, 1+t)) = \beta(\delta_t(x), 1+t)$. Extend F_0 to F, a homeomorphism of E^n, by $F(0) = 0$ and $F(\beta(x, r)) = \gamma^{-1}h^q\gamma F_0(\beta(x, 2^q r))$ where q is the unique integer such that $1 < 2^q r \leqq 2$. One easily sees that $F^{-1}\gamma h \gamma^{-1} F(\beta(x, r)) = \beta(x, \frac{1}{2}r)$.

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