

A MODEL OF EUCLIDEAN 2-SPACE

M. S. KRISHNA SASTRY¹

ABSTRACT. In this paper a model of Euclidean 2-space, called the spin model, is introduced. To each complex-valued function f defined in an open subset of the complex plane is associated a function \tilde{f} mapping an open subset of the spin model space into the two-dimensional real vector space of two-rowed real column matrices. Cauchy's theorem and Cauchy's integral formula for an analytic function f are written as theorems involving the function \tilde{f} .

1. Introduction. The purpose of the present note is to introduce a model of Euclidean 2-space, which we will call 'The spin model of Euclidean 2-space,' and to rewrite Cauchy's theorem and Cauchy integral formula as results concerning a function from a subset of the spin model of Euclidean 2-space into a two-dimensional real vector space. These results have suggested a method of defining a concept of analyticity for functions from a nonempty open subset of the spin model of Euclidean 3-space [1] into a two-dimensional complex vector space and also the forms of the analogs of Cauchy's theorem and Cauchy integral formula in this set up. In a subsequent paper we will define this concept of analyticity and show that these functions do have some properties similar to the properties of (ordinary) analytic functions.

2. Let E_2 denote the abstract two-dimensional Euclidean space. We identify E_2 with two-rowed real column vectors. The scalar product in E_2 , denoted (\cdot, \cdot) , is the usual one, namely: if $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ are any two elements of E_2 then $(x, y) = x_1y_1 + x_2y_2$.

By \mathfrak{C}_2 we denote the set of all symmetric linear transformations of E_2 into itself of trace zero. Relative to an orthonormal (o.n.) basis in E_2 each element $T \in \mathfrak{C}_2$ has a matrix representation:

$$T \leftrightarrow \begin{pmatrix} x_2 & x_1 \\ x_1 & -x_2 \end{pmatrix} \quad \text{for some } x_1, x_2 \in \mathbf{R}.$$

Received by the editors February 7, 1969.

AMS 1970 subject classifications. Primary 30-XX, 30A92, 30A96; Secondary 30A96.

Key words and phrases. Spin model of Euclidean 2-space, Fréchet differentiable, Cauchy-Riemann operator.

¹ These results have appeared in the author's Ph.D. thesis submitted to the University of Rochester, Rochester, New York under the direction of Professor W. F. Eberlein.

Copyright © 1971, American Mathematical Society

Let

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $T = x_1\tau_1 + x_2\tau_2$ and clearly $\{\tau_1, \tau_2\}$ is a basis over \mathbf{R} for \mathfrak{C}_2 . Thus \mathfrak{C}_2 is a two-dimensional real vector space.

Let $A \in \mathfrak{C}_2$. Then by properly choosing an o.n. basis in E_2 we can let

$$A \leftrightarrow \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$$

for some $a \in \mathbf{R}$.

Hence $A^2 = a^2I$, where I is the 2×2 identity matrix. From this it immediately follows that if $A, B \in \mathfrak{C}_2$ then $AB + BA = kI$, for some $k \in \mathbf{R}$.

DEFINITION. Let $A, B \in \mathfrak{C}_2$. Then the scalar product of A and B , denoted $A \cdot B$, is defined by $AB + BA = 2(A \cdot B)I$.

It can be easily verified that this definition satisfies all the requirements for a scalar product and that it is nondegenerate.

Thus \mathfrak{C}_2 with this scalar product is a two-dimensional Euclidean space and we call \mathfrak{C}_2 , following Eberlein [1], the spin model of Euclidean 2-space.

3. Let $\{\rho_1, \rho_2\}$ be an o.n. basis for \mathfrak{C}_2 and let f be a smooth mapping from an open subset of \mathfrak{C}_2 into E_2 , where we are endowing \mathfrak{C}_2 with its natural topology. For $k = 1, 2$, let

$$\frac{\partial f}{\partial x_k} = \lim_{h \rightarrow 0} \frac{f(x + h\rho_k) - f(x)}{h} \equiv \partial_k f.$$

DEFINITION. $\nabla = \rho_1\partial_1 + \rho_2\partial_2$.

We observe that ∇ can act on smooth functions from open subsets of \mathfrak{C}_2 into E_2 —e.g. if f is as above then

$$\nabla f = \rho_1\partial_1 f + \rho_2\partial_2 f.$$

The proof that the definition of ∇ is independent of the choice of the o.n. basis in \mathfrak{C}_2 is similar to the corresponding result in [1].

If we choose $\{\tau_1, \tau_2\}$ as an o.n. basis for \mathfrak{C}_2 we can write ∇ as a matrix operator, namely

$$\nabla = \begin{pmatrix} \partial/\partial x_2 & \partial/\partial x_1 \\ \partial/\partial x_1 & -\partial/\partial x_2 \end{pmatrix},$$

and this is precisely the Cauchy-Riemann operator considered in [2].

4. Definition. Let E and F be finite-dimensional vector spaces over \mathbb{R} and let \mathcal{E} and F have their natural topologies. Let S be a nonempty open subset of E . Let $f: S \rightarrow F$. f is said to be Fréchet differentiable at $x \in S$ iff there exists a linear transformation $f'(x): E \rightarrow F$ such that

$$f(x + y) = f(x) + f'(x)y + \|y\|\delta(x, y),$$

where $\delta(x, y) \rightarrow 0$ as $y \rightarrow 0$. f is Fréchet differentiable in S if f is Fréchet differentiable at each point of S .

For each subset D of \mathbb{C} (= the set of complex numbers) let

$$\tilde{D} = \{x_1\tau_1 + x_2\tau_2 : (x_1, x_2) \in D\}.$$

Then $\tilde{D} \subset E_2$ and D and \tilde{D} are homeomorphic.

Let $f: D (\subset \mathbb{C}) \rightarrow \mathbb{C}$, with

$$f((x_1, x_2)) = u(x_1, x_2) + iv(x_1, x_2) \quad (i = \sqrt{-1}).$$

Then we let $\tilde{f}: \tilde{D} \rightarrow E_2$ with

$$\tilde{f}(x_1\tau_1 + x_2\tau_2) = \begin{pmatrix} u(x_1, x_2) \\ v(x_1, x_2) \end{pmatrix}.$$

Let f and D be as above and suppose D is open. We know that (see pp. 55–59 of [3] and in particular §6.4 on p. 59) f is analytic in D iff the mapping $(x_1, x_2) \rightarrow (u, v)$ is Fréchet differentiable in D and the partial derivatives of u and v satisfy the Cauchy-Riemann equations. Observing that \tilde{f} belongs to the null space of ∇ iff u and v satisfy the Cauchy-Riemann equations, we conclude that f is analytic in D iff \tilde{f} is Fréchet differentiable in \tilde{D} and $\nabla\tilde{f} = 0$ in \tilde{D} .

Hence it appears that some of the results concerning an analytic function f can be formulated equivalently in terms of the function \tilde{f} .

The rest of the paper is devoted to rewriting Cauchy's theorem and Cauchy integral formula as theorems concerning functions mapping subsets of \mathcal{E}_2 into E_2 .

We conclude this section with a couple of definitions.

DEFINITION. Let \tilde{f} be a continuous map from an open subset \tilde{D} of \mathcal{E}_2 into E_2 . Let μ be a Borel measure on \mathcal{E}_2 . If E is a bounded subset of \tilde{D} then

$$\int_E \tilde{f} d\mu = \begin{pmatrix} \int_E u d\mu \\ \int_E v d\mu \end{pmatrix}.$$

DEFINITION. Let $n: \mathbb{C}_2 \rightarrow \mathbb{C}_2$ and let $\bar{f}: \bar{D} (C \mathbb{C}_2) \rightarrow E_2$. Then $n\bar{f}: \bar{D} \rightarrow E_2$ is defined by

$$(n\bar{f})(x) = n(x)\bar{f}(x) \quad (x \in \bar{D})$$

where the multiplication on the right side is the usual matrix multiplication.

5. Let $(x_1, x_2) = z \in \mathbb{C}$, where x_1, x_2 are real numbers. We define

$$z_m = \begin{pmatrix} x_2 & x_1 \\ x_1 & -x_2 \end{pmatrix}, \quad z_c = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad z^\perp = (-x_2, x_1).$$

Notice that $z^\perp = iz$, where $i^2 = -1$. It is easily verified that if $z_1, z_2 \in \mathbb{C}$, then

$$(*) \quad (\overline{z_1 z_2})_c = (z_1^\perp)_m (z_2)_c,$$

where the multiplication on the left is the multiplication of two complex numbers, the multiplication on the right is the matrix multiplication and \bar{z} denotes the complex conjugate of z .

Cauchy's theorem states that if f is analytic in a simply connected region $D \subset \mathbb{C}$ and if Γ is a simple closed curve in D then

$$\int_\Gamma f(z) dz = 0 \quad \text{i.e.} \quad \int_\Gamma f(z) \frac{dz}{ds} ds = 0,$$

where ds denotes the elemental arc length. Hence the conclusion of Cauchy's theorem can be stated equivalently as

$$\int_\Gamma (\overline{dz/ds}) f(z) ds = 0.$$

Hence, on identifying D and \bar{D} and using $(*)$ we can write the conclusion of Cauchy's theorem in the equivalent form

$$\int_\Gamma \left(- \left(\frac{dz}{ds} \right)^\perp \right)_m \bar{f}(z) ds = 0.$$

Since dz/ds is the unit tangent vector at z to Γ , $-(dz/ds)^\perp$ will be the outward unit normal vector at z to Γ . Thus Cauchy's theorem can be restated as follows:

Let D be a simply connected region in \mathbb{C} and let Γ be a simple closed curve in D . Let f be analytic in D . Then

$$\int_\Gamma (n\bar{f})(z) ds = 0,$$

where n denotes the outward unit normal vector at z to Γ and where we identified D and \tilde{D} .

DEFINITION. Let $p \in \mathbb{E}_2$. We define $g(\cdot, p): \mathbb{E}_2 - \{p\} \rightarrow \mathbb{E}_2$ by $g(x, p) = \text{grad}(-\log(1/r))$, where 'grad' denotes gradient and $r = \sqrt{((x-p) \cdot (x-p))}$. If, relative to an o.n. basis in \mathbb{E}_2 , $x = (x_1, x_2)$ and $p = (p_1, p_2)$, then

$$g(x, p) = r^{-2}(x_1 - p_1, x_2 - p_2).$$

Cauchy integral formula states that if Γ is a simple closed curve in a region D of analyticity of a function f and if p is a point in the interior of Γ then

$$f(p) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - p} dz.$$

It can easily be shown that the above result is equivalent to

$$\tilde{f}(p) = \frac{1}{2\pi} \int_{\Gamma} (g(z, p))_m (n\tilde{f})(z) ds,$$

where, again, we identify D and \tilde{D} and n is the outward unit normal vector at z to Γ .

We wish to point out that the expression $-\log(1/r)$ which appeared above must be interpreted as the fundamental solution of the two-dimensional Laplace equation. This interpretation will suggest the form of the analog of Cauchy integral formula for functions mapping subsets of the spin model of Euclidean 3-space into a two-dimensional complex vector space.

REFERENCES

1. W. F. Eberlein, *The spin model of Euclidean 3-space*, Amer. Math. Monthly **69** (1962), 587-598.
2. ———, *Cauchy-Riemann operator*, Amer. J. Phys. **35** (1967), 53.
3. S. Saks and A. Zygmund, *Analytic functions*, 2nd ed., Monografie Mat., Tom 28, PWN, Warsaw, 1965. MR 31 #4889.

OHIO UNIVERSITY, ATHENS, OHIO 45701