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A MODEL OF EUCLIDEAN 2-SPACE

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ABSTRACT. In this paper a model of Euclidean 2-space, called the spin model, is introduced. To each complex-valued function f defined in an open subset of the complex plane is associated a function $\tilde{f}$ mapping an open subset of the spin model space into the two-dimensional real vector space of two-rowed real column matrices. Cauchy's theorem and Cauchy's integral formula for an analytic function f are written as theorems involving the function $\tilde{f}$.

1. **Introduction.** The purpose of the present note is to introduce a model of Euclidean 2-space, which we will call 'The spin model of Euclidean 2-space,' and to rewrite Cauchy's theorem and Cauchy integral formula as results concerning a function from a subset of the spin model of Euclidean 2-space into a two-dimensional real vector space. These results have suggested a method of defining a concept of analyticity for functions from a nonempty open subset of the spin model of Euclidean 3-space [1] into a two-dimensional complex vector space and also the forms of the analogs of Cauchy's theorem and Cauchy integral formula in this set up. In a subsequent paper we will define this concept of analyticity and show that these functions do have some properties similar to the properties of (ordinary) analytic functions.

2. Let E_2 denote the abstract two-dimensional Euclidean space. We identify E_2 with two-rowed real column vectors. The scalar product in E_2, denoted $(\cdot,\ \cdot)$, is the usual one, namely: if $x=\binom{x_1}{x_2}$ and $y=\binom{y_1}{y_2}$ are any two elements of E_2 then $(x, y)=x_1y_1+x_2y_2$.

By $\mathfrak{E}_2$ we denote the set of all symmetric linear transformations of E_2 into itself of trace zero. Relative to an orthonormal (o.n.) basis in E_2 each element $T\in E_2$ has a matrix representation:

$$T \leftrightarrow \begin{pmatrix} x_2 & x_1 \\ x_1 & -x_2 \end{pmatrix} \qquad \text{for some } x_1, x_2 \in \mathbf{R}.$$

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Let

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $T = x_1\tau_1 + x_2\tau_2$ and clearly $\{\tau_1, \tau_2\}$ is a basis over $\mathbf{R}$ for $\mathfrak{S}_2$. Thus $\mathfrak{S}_2$ is a two-dimensional real vector space.

Let $A \in \mathfrak{S}_2$. Then by properly choosing an o.n. basis in E_2 we can let

$$A \leftrightarrow \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$$

for some $a \in \mathbf{R}$.

Hence $A^2 = a^2 I$, where I is the 2×2 identity matrix. From this it immediately follows that if $A, B \in \mathfrak{S}_2$ then $AB + BA = kI$, for some $k \in \mathbf{R}$.

DEFINITION. Let $A, B \in \mathfrak{S}_2$. Then the scalar product of A and B, denoted $A \cdot B$, is defined by $AB + BA = 2(A \cdot B)I$.

It can be easily verified that this definition satisfies all the requirements for a scalar product and that it is nondegenerate.

Thus $\mathfrak{S}_2$ with this scalar product is a two-dimensional Euclidean space and we call $\mathfrak{S}_2$, following Eberlein [1], the spin model of Euclidean 2-space.

3. Let $\{\rho_1, \rho_2\}$ be an o.n. basis for $\mathfrak{S}_2$ and let f be a smooth mapping from an open subset of $\mathfrak{S}_2$ into E_2, where we are endowing $\mathfrak{S}_2$ with its natural topology. For $k = 1, 2$, let

$$\frac{\partial f}{\partial x_k} = \lim_{h \to 0} \frac{f(x + h\rho_k) - f(x)}{h} \equiv \partial_k f.$$

DEFINITION. $\nabla = \rho_1\partial_1 + \rho_2\partial_2$.

We observe that ∇ can act on smooth functions from open subsets of $\mathfrak{S}_2$ into E_2—e.g. if f is as above then

$$\nabla f = \rho_1\partial_1 f + \rho_2\partial_2 f.$$

The proof that the definition of ∇ is independent of the choice of the o.n. basis in $\mathfrak{S}_2$ is similar to the corresponding result in [1].

If we choose $\{\tau_1, \tau_2\}$ as an o.n. basis for $\mathfrak{S}_2$ we can write ∇ as a matrix operator, namely

$$\nabla = \begin{pmatrix} \partial/\partial x_2 & \partial/\partial x_1 \\ \partial/\partial x_1 & -\partial/\partial x_2 \end{pmatrix},$$

and this is precisely the Cauchy-Riemann operator considered in [2].

4. **Definition.** Let E and F be finite-dimensional vector spaces over $\mathbf{R}$ and let $\mathfrak{E}$ and F have their natural topologies. Let S be a nonempty open subset of E. Let $f: S \to F$. f is said to be Fréchet differentiable at $x \in S$ iff there exists a linear transformation $f'(x): E \to F$ such that

$$f(x+y) = f(x) + f'(x)y + \|y\| \delta(x, y),$$

where $\delta(x, y) \to 0$ as $y \to 0$. f is Fréchet differentiable in S if f is Fréchet differentiable at each point of S.

For each subset D of $\mathbf{C}$ (= the set of complex numbers) let

$$\tilde{D} = \{x_1\tau_1 + x_2\tau_2 : (x_1, x_2) \in D\}.$$

Then $\tilde{D} \subset E_2$ and D and $\tilde{D}$ are homeomorphic.

Let $f: D\ (\subset \mathbf{C}) \to \mathbf{C}$, with

$$f((x_1, x_2)) = u(x_1, x_2) + iv(x_1, x_2) \qquad (i = \sqrt{(-1)}).$$

Then we let $\tilde{f}: \tilde{D} \to E_2$ with

$$\tilde{f}(x_1\tau_1 + x_2\tau_2) = \begin{pmatrix} u(x_1, x_2) \\ v(x_1, x_2) \end{pmatrix}.$$

Let f and D be as above and suppose D is open. We know that (see pp. 55–59 of [3] and in particular §6.4 on p. 59) f is analytic in D iff the mapping $(x_1, x_2) \to (u, v)$ is Fréchet differentiable in D and the partial derivatives of u and v satisfy the Cauchy-Riemann equations. Observing that $\tilde{f}$ belongs to the null space of ∇ iff u and v satisfy the Cauchy-Riemann equations, we conclude that f is analytic in D iff $\tilde{f}$ is Fréchet differentiable in $\tilde{D}$ and $\nabla\tilde{f} = 0$ in $\tilde{D}$.

Hence it appears that some of the results concerning an analytic function f can be formulated equivalently in terms of the function $\tilde{f}$.

The rest of the paper is devoted to rewriting Cauchy's theorem and Cauchy integral formula as theorems concerning functions mapping subsets of $\mathfrak{E}_2$ into E_2.

We conclude this section with a couple of definitions.

DEFINITION. Let $\tilde{f}$ be a continuous map from an open subset $\tilde{D}$ of $\mathfrak{E}_2$ into E_2. Let μ be a Borel measure on $\mathfrak{E}_2$. If E is a bounded subset of $\tilde{D}$ then

$$\int_E \tilde{f} d\mu = \begin{pmatrix} \int_E u d\mu \\ \int_E v d\mu \end{pmatrix}.$$

DEFINITION. Let $n:\mathfrak{E}_2\to\mathfrak{E}_2$ and let $\bar{f}:\tilde{D}\ (\subset\mathfrak{E}_2)\to E_2$. Then $n\bar{f}:\tilde{D}\to E_2$ is defined by

$$(n\bar{f})(x), = n(x)\bar{f}(x) \qquad (x\in\tilde{D})$$

where the multiplication on the right side is the usual matrix multiplication.

5. Let $(x_1, x_2)=z\in\mathbf{C}$, where x_1, x_2 are real numbers. We define

$$z_m, = \begin{pmatrix} x_2 & x_1 \\ x_1 & -x_2 \end{pmatrix}, \qquad z_c =, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad z^\perp = (-x_2, x_1).$$

Notice that $z^\perp = iz$, where $i^2=-1$. It is easily verified that if $z_1, z_2\in\mathbf{C}$, then

$$(*) \qquad (\overline{z_1z_2})_c = (z_1^\perp)_m(z_2)_c,$$

where the multiplication on the left is the multiplication of two complex numbers, the multiplication on the right is the matrix multiplication and $\bar{z}$ denotes the complex conjugate of z.

Cauchy's theorem states that if f is analytic in a simply connected region $D\subset\mathbf{C}$ and if Γ is a simple closed curve in D then

$$\int_\Gamma f(z)dz = 0 \quad \text{i.e.} \quad \int_\Gamma f(z)\frac{dz}{ds}\,ds = 0,$$

where ds denotes the elemental arc length. Hence the conclusion of Cauchy's theorem can be stated equivalently as

$$\int_\Gamma \overline{(dz/ds)f(z)}\,ds = 0.$$

Hence, on identifying D and $\tilde{D}$ and using $(*)$ we can write the conclusion of Cauchy's theorem in the equivalent form

$$\int_\Gamma \left(-\left(\frac{dz}{ds}\right)^\perp\right)_m \bar{f}(z)ds = 0.$$

Since dz/ds is the unit tangent vector at z to Γ, $-(dz/ds)^\perp$ will be the outward unit normal vector at z to Γ. Thus Cauchy's theorem can be restated as follows:

Let D be a simply connected region in $\mathbf{C}$ and let Γ be a simple closed curve in D. Let f be analytic in D. Then

$$\int_\Gamma (n\bar{f})(z)ds = 0,$$

where n denotes the outward unit normal vector at z to Γ and where we identified D and $\tilde{D}$.

DEFINITION. Let $p \in \mathfrak{E}_2$. We define $g(\cdot,\ p):\mathfrak{E}_2 - \{p\} \to \mathfrak{E}_2$ by $g(x,\ p) = \text{grad}\ (-\log\ (1/r))$, where 'grad' denotes gradient and $r = \sqrt{((x-p)\cdot(x-p))}$. If, relative to an o.n. basis in $\mathfrak{E}_2$, $x = (x_1,\ x_2)$ and $p = (p_1, p_2)$, then

$$g(x, p) = r^{-2}(x_1 - p_1, x_2 - p_2).$$

Cauchy integral formula states that if Γ is a simple closed curve in a region D of analyticity of a function f and if p is a point in the interior of Γ then

$$f(p) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - p}\, dz.$$

It can easily be shown that the above result is equivalent to

$$\bar{f}(p) = \frac{1}{2\pi}\int_\Gamma (g(z,\ p))_m(n\bar{f})(z)ds,$$

where, again, we identify D and $\tilde{D}$ and n is the outward unit normal vector at z to Γ.

We wish to point out that the expression $-\log(1/r)$ which appeared above must be interpreted as the fundamental solution of the two-dimensional Laplace equation. This interpretation will suggest the form of the analog of Cauchy integral formula for functions mapping subsets of the spin model of Euclidean 3-space into a two-dimensional complex vector space.

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