PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, No. 1, April 1971

ALMOST PERIODIC SOLUTIONS OF POISSON'S EQUATION[1]

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ABSTRACT. It is well known that if $f(t)$ is almost periodic with respect to a real variable t, and if $F(t)=\int^t f(s)ds$ is bounded for $-\infty<t<\infty$, then $F(t)$ is also almost periodic in t. We shall prove a similar result for Poisson's equation.

1. Introduction. In this report, we shall prove the following theorem.

THEOREM. *Let $f(P)$ be an almost periodic function of P in R^m, and let $u(P)$ be a bounded continuous function of P in R^m. Assume that $u(P)$ is a solution of*

$$\Delta u = f \qquad (\Delta = \text{div}\cdot\text{grad, } \textit{i.e. Laplacian}) \tag{1.1}$$

in the sense of distribution. Then $u(P)$ is almost periodic with respect to P in R^m.

A function $f(P)$ is almost periodic with respect to P in R^m when
(i) f is continuous in R^m;
(ii) for every sequence $\{P_n\}$ in R^m, the corresponding sequence $\{f(P+P_n)\}$ contains a subsequence which is convergent uniformly in R^m.

This definition is equivalent to the definition based on the condition that the set of almost periods of f is relatively dense in R^m. If f is almost periodic with respect to P in R^m, then $f(P)$ is bounded in R^m.

A function $u(P)$ is a solution of (1.1) in the sense of distribution, when

$$\int_{R^m} u(P)\Delta\phi(P)dP = \int_{R^m} f(P)\phi(P)dP$$

Received by the editors June 9, 1970.

AMS 1969 *subject classifications.* Primary 3135, 3514, 3575, 4230; Secondary 3520, 3539, 4220, 4640, 4765, 7635.

Key words and phrases. Poisson's equation, bounded generalized solutions, almost periodicity in R^m, almost periodic solutions.

[1] This research was supported in part by the National Science Foundation under Grant Number GP7041X. The problem of this paper was given to the author by Professor Daniel D. Joseph, Aeronautical Engineering and Mechanics Department, University of Minnesota. The author wishes to express his appreciation to Professor Hans Weinberger for his advice during the study of this problem.

for every C^∞-function ϕ of P with compact support. Since $f(P)$ and $u(P)$ are bounded in R^m, there exists a positive constant K such that

$$|u(P) - u(Q)| \leqq K|P - Q| \tag{1.2}$$

for P and Q in R^m, where $|P-Q|$ denotes the distance between P and Q [1, Chapter IV].

2. **Preliminaries.** Let $f(t)$ be an almost periodic function of one real variable t. It is known that, if $\int_0^t f(s)ds$ is bounded for $t \in R^1$, then $\int_0^t f(s)ds$ is almost periodic with respect to t. We shall prove our theorem in a manner similar to J. Favard's proof of the statement given above [2, pp. 82–85]. To do this, we shall need the estimate (1.2) and the following fact:

Let $u_1(P)$ and $u_2(P)$ be bounded continuous functions of P in R^m, and assume that u_1 and u_2 are solutions of $\Delta u = f$ in the sense of distribution. Then $u_1(P) - u_2(P)$ is identically equal to a constant, since it is a harmonic function which is bounded in R^m. Hence if

$$\inf_P u_1(P) = \inf_P u_2(P), \tag{2.1}$$

then $u_1(P)$ is identically equal to $u_2(P)$.

3. **Proof of Theorem: Part I.** Since f is almost periodic and u satisfies the estimate (1.2), it is sufficient to prove that, if $\{P_n\}$ is a sequence of points in R^m such that

$$\lim_{n\to\infty} f(P + P_n) = g(P) \tag{3.1}$$

uniformly in R^m, and

$$\lim_{n\to\infty} u(P + P_n) = v(P) \tag{3.2}$$

uniformly on every compact set in R^m, then

$$\lim_{n\to\infty} u(P + P_n) = v(P) \quad \textit{uniformly in } R^m. \tag{3.3}$$

First of all, we shall prove that

$$\inf v(P) = \inf u(P). \tag{3.4}$$

It is evident that v is a solution of $\Delta v = g$ in the sense of distribution and that v is a bounded continuous function of P in R^m. On the other hand, (3.1) implies $\lim_{n\to\infty} g(P - P_n) = f(P)$ uniformly in R^m. Hence if we choose a subsequence $\{n_\nu\}$ of $\{n\}$ so that $\lim_{\nu\to\infty} v(P - P_{n_\nu}) = w(P)$ uniformly on each compact set in R^m, then w is a bounded continuous function and a solution of $\Delta w = f$ in the sense of distribution. Therefore we get $w(P) = u(P) + C$, where C is a constant. Observe that we have inf $u \leqq$ inf $v \leqq$ inf w and sup $u \geqq$ sup $v \geqq$ sup w. Hence $C = 0$, and we get (3.4).

4. **Proof of Theorem: Part II.** Assume that (3.3) is not true. Then there exist a positive constant α and a sequence $\{Q_{n_j}\}$ such that

$$(4.1) \qquad \left| u(Q_{n_j} + P_{n_j}) - v(Q_{n_j}) \right| \geqq \alpha.$$

Choose a subsequence $\{n_\nu\}$ of $\{n_j\}$ so that

$$(4.2) \qquad \lim_{\nu \to \infty} f(P + Q_{n_\nu} + P_{n_\nu}) = h(P)$$

uniformly in R^m and

$$(4.3) \qquad \lim_{\nu \to \infty} u(P + Q_{n_\nu} + P_{n_\nu}) = w_1(P), \qquad \lim_{\nu \to \infty} v(P + Q_{n_\nu}) = w_2(P)$$

uniformly on each compact set in R^m. Then (3.1) and (4.2) imply that

$$(4.4) \qquad \lim_{\nu \to \infty} g(P + Q_{n_\nu}) = h(P)$$

uniformly in R^m. Hence w_1 and w_2 are two bounded continuous functions which are solutions of $\Delta w = h$ in the sense of distribution. On the other hand, inf $w_1 =$ inf $u =$ inf $v =$ inf w_2. Hence w_1 must be identically equal to w_2. However, this is impossible, since (4.1) and (4.3) imply that

$$\left| w_1(0) - w_2(0) \right| \geqq \alpha \geqq 0.$$

This completes the proof of our theorem.

5. **Remarks.** As we mentioned in §2, we used two properties of Laplacian in the proof of our theorem:

(i) inequality (1.2);

(ii) a harmonic function which is bounded in R^m is identically equal to a constant.

Therefore, if we want to replace Laplacian by another partial differential operator A with constant coefficients, it might be helpful to know when an operator A will enjoy two properties similar to (i) and (ii).

Property (i) is closely related to the translation invariant property of Laplacian and to the existence of Green's function.

Property (ii) may be replaced by the property that every bounded solution of $Au = 0$ is almost periodic in R^m. In cases where property

(i) is satisfied and no nontrivial bounded solution of $Au=0$ approaches zero, it might be possible to show that, if there exist bounded solutions of $Au=f(P)$, where $f(P)$ is almost periodic, then at least one of them is almost periodic in R^m. There is such a theorem for ordinary differential equations with variable coefficients [2, p. 95].

Finally, it might be interesting to know that

$$\Delta u + u = 0 \qquad (\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2)$$

is satisfied by $u(x, y) = J_0(\sqrt{(x^2+y^2)})$, where J_0 is the Bessel function of the first kind of order 0. It is evident that $u(x, y)$ is bounded and $\lim_{r\to\infty} J_0(r) = 0$.

References

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