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KOECHER'S PRINCIPLE FOR QUADRATIC JORDAN ALGEBRAS

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ABSTRACT. In this note we indicate two techniques for establishing identities in quadratic Jordan algebras. The first method, due to Professor M. Koecher, shows that to establish an identity in general it suffices to establish it when all the elements involved are invertible. The second technique involves interpreting a given identity in a Jordan algebra as a simpler identity in a homotope of that algebra. These two techniques are applied to derive some important identities.

Throughout we use the conventions of [3] regarding quadratic Jordan algebras over an arbitrary ring of scalars Φ. Thus we have a product $U_x y$ linear in y and quadratic in x satifying

(UQJ I) $\quad U_1 = I \qquad$ (1 the unit),

(UQJ II) $\quad U_{U(x)y} = U_x U_y U_x,$

(UQJ III) $\quad U_x V_{y,x} = V_{x,y} U_x \qquad (V_{x,y} z = \{x\ y\ z\} = U_{x,z} y).$

It is common experience that various identities which appear in the theory of quadratic Jordan algebras require rather cumbersome proofs, so any principle which makes it easier to verify identities provides a welcome labor-saving device. The most celebrated of these is *Macdonald's Principle*, which says that any identity in three variables x, y, z of degree $\leqq 1$ in z will be valid in all Jordan algebras if it is valid in all special Jordan algebras [2, p. 41], [4]. (It is a simple matter to check whether an identity holds in special algebras.)

The *Koecher Principle* says that if an identity in any number of variables $x_1, \cdots, x_n$ is satisfied whenever the x_i are invertible, then it is satisfied in general. More generally,

KOECHER'S PRINCIPLE. *To establish that a Φ-identity $f(x_1, \cdots, x_n) = 0$ holds for all quadratic Jordan Φ-algebras it suffices to establish it only when*

(i) $\mathfrak{J}$ *is finitely spanned as a Φ-module,*

(ii) $\mathfrak{J} = \Gamma 1 + \mathfrak{N}$ *where $\mathfrak{N}$ is nilpotent and Γ the centroid,*

(iii) $x_i = \lambda_i 1 + z_i$ *for $\alpha_i \in \Gamma$ invertible and $z_i \in \mathfrak{N}$.*

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PROOF. A Φ-identity $f(x_1, \cdots, x_n)$ is an element of the free quadratic Jordan algebra $\mathfrak{F}=\Phi[x_1, \cdots, x_n]$ (e.g. see [4]); $f(x_1, \cdots, x_n) =0$ holds identically for all Φ-algebras if and only if f is the zero element of $\mathfrak{F}$. Suppose the total degree of f in all its variables is m (we count $U_x y$ as being of degree 2 in x). Then f is zero in $\mathfrak{F}$ if and only if it is zero in $\overline{\mathfrak{F}}=\mathfrak{F}/K$ for K the ideal spanned by all monomials of degree $>m$. (This is the basic idea which makes things work; we can forget about all but a finite part of $\mathfrak{F}$.)

Note that $\overline{\mathfrak{F}}$ is finitely spanned since there are only finitely many products of degree $\leqq m$ involving only the variables $\bar{x}_1, \cdots, \bar{x}_n$. Also, if $\mathfrak{M}$ is the ideal spanned by all monomials of degree $\geqq 1$ in $\mathfrak{F}$ then $\overline{\mathfrak{F}}=\Phi\bar{1}+\overline{\mathfrak{M}}$ where $\overline{\mathfrak{M}}^{m+1}=0$ since $\mathfrak{M}^{m+1}\subset\mathfrak{K}$.

If we take $\Omega=\Phi[\lambda_1, \cdots, \lambda_n]$ where the λ_i are independent commuting elements satisfying $\lambda_i^{m+1}=1$ (so Ω is the group algebra of $G_1\times \cdots \times G_n$ for G_i cyclic of order $m+1$ with generator λ_i), then Ω is free over Φ with basis the $\lambda_1^{e_1}\cdots\lambda_n^{e_n}$ for $0\leqq e_i\leqq m$. Thus $\overline{\mathfrak{F}}$ is imbedded in $\mathfrak{J}=\overline{\mathfrak{F}}_\Omega$. $\overline{\mathfrak{F}}$ is still finitely spanned over Φ since Ω is, and $\mathfrak{J}=\Gamma 1+\mathfrak{N}$ since $\mathfrak{J}_\Omega=\Omega 1+\overline{\mathfrak{M}}_\Omega$ where Ω is contained in the centroid and $\mathfrak{N}=\overline{\mathfrak{M}}_\Omega$ still satisfies $\mathfrak{N}^{m+1}=0$. The $\lambda_i\in\Omega$ are invertible and $z_i=\bar{x}_i$ are in $\mathfrak{N}$.

Thus $\mathfrak{J}$ and the $\lambda_i 1+z_i$ satisfy the hypothesis of the Principle, so

$$f(\lambda_1 1 + z_1, \cdots, \lambda_n 1 + z_n) = 0.$$

Expanding this out and identifying coefficients of $\lambda_1^{e_1}\cdots\lambda_n^{e_n}$ (noting that the λ_i appear only to powers $e_i\leqq m$ since f is of degree at most m in any variable, so the various $\lambda_1^{e_1}\cdots\lambda_n^{e_n}$ are distinct), the constant term is just

$$f(z_1, \cdots, z_n) = 0.$$

This means

$$f(\bar{x}_1, \cdots, \bar{x}_n) = \overline{f(x_1, \cdots, x_n)}$$

is zero in $\overline{\mathfrak{F}}_\Omega$, hence in $\overline{\mathfrak{F}}$ as $\overline{\mathfrak{F}}$ is imbedded in $\overline{\mathfrak{F}}_\Omega$. We noted before that this implied $f(x_1, \cdots, x_n)$ was zero in $\mathfrak{F}$.

Certainly if we can establish an identity whenever (iii) holds then we can establish whenever (i), (ii), (iii) hold simultaneously. Again, if we can establish it when the x_i are invertible we can establish it when (iii) holds, since $\lambda_i 1+z_i$ is invertible for invertible α_i and nilpotent z_i.

COROLLARY 1. *If a Φ-identity $f(x_1, \cdots, x_n)=0$ holds for all Jordan Φ-algebras when the x_i are invertible elements then it holds when the x_i are arbitrary.*

Similarly, it is more than enough to establish an identity whenever (i) holds. If Ω is a free extension of Φ then any Φ-algebra $\mathfrak{J}$ is imbedded in an Ω-algebra $\mathfrak{J}_\Omega$, so if f vanishes on all Ω-algebras it vanishes on all Φ-algebras. Applying the Principle to Ω we obtain

COROLLARY 2 [2, p. 423]. *If Φ is a field, a Φ-identity $f(x_1, \cdots, x_n) = 0$ holds for all Jordan Φ-algebras if there is a field extension Ω of Φ such that $f(x_1, \cdots, x_n) = 0$ holds for all finite-dimensional Ω-algebras.*

Usually one takes Ω to be infinite (in case Φ is finite) so that the Zariski topology and the differential calculus can be used. Here it suffices to prove $f(x_1, \cdots, x_n) = 0$ on a Zariski-dense subset, so we may restrict the x_i to lie in the complement of some algebraic set. (For example, requiring the generic norm to be nonzero, $N(x_i) \neq 0$, is just the condition that the x_i be invertible.)

REMARK. A similar principle holds for any variety of linear algebras defined by homogeneous identities (so that the free algebra is graded in the natural way) and admitting adjunction of a unit and scalar extensions. Examples are alternative, commutative Jordan, noncommutative Jordan, and strictly power-associative algebras.

2. **Applications.** We now want to apply the principle to establish certain identities needed in a forthcoming paper [5]. Recall that if $u \in \mathfrak{J}$ we can define a new Jordan algebra $\mathfrak{J}^{(u)}$, the *u-homotope* of $\mathfrak{J}$, having the same linear structure but multiplication

$$U_x^{(u)} = U_x U_u, \qquad x^{2(u)} = U_x u$$

and derived operations

$$V_x^{(u)} = V_{x,u}, \qquad V_{x,y}^{(u)} = V_{x,U(u)y}.$$

If u is invertible then $\mathfrak{J}^{(u)}$ will have unit $1^{(u)} = u^{-1}$, and we call $\mathfrak{J}^{(u)}$ the *u-isotope* of $\mathfrak{J}$.

Note that in passing from $\mathfrak{J}$ to $\mathfrak{J}^{(u)}$ our operators "expand", so that in going from $\mathfrak{J}^{(u)}$ to $\mathfrak{J}$ they "shrink". Thus the fundamental formula $U_{U(x)y} = U_x U_y U_x$ in $\mathfrak{J}$ becomes, when multiplied on the right by U_y, just $U^{(y)}_{x^{2(y)}} = U_x^{(y)} U_x^{(y)}$, i.e. the relation $U_{x^2} = U_x^2$ in the homotope $\mathfrak{J}^{(y)}$. Thus for invertible y the task of proving the fundamental formula of degree 7 reduces to establishing a formula of degree 5. In other words, *we interpret our given identity in $\mathfrak{J}$ as some simpler identity in a homotope $\mathfrak{J}^{(y)}$.* We use this technique to establish:

(1) if $p\mathfrak{J} = 0$ for some prime p then $V_{x,y}^p = V_{z,y}$ for $z = x^{p(y)}$;

(2) $V_{x,y} V_{z,y} = V_{x,U(y)z} + U_{x,z} U_y$;

(3) $V_{x,y}V_{x,z}=V_{U(x)y,z}+U_xU_{y,z}$;

(4) $V_{x,y}U_z+U_zV_{y,x}=U_{V(x,y)z,z}$;

and, if we define $T_{x,y}=I-V_{x,y}+U_xU_y$,

(5) $T_{x,y}T_{-x,y}=T_{x,y}T_{x,-y}=T_{x,U(y)x}=T_{U(x)y,y}$;

(6) $T_{x,y}U_zT_{y,x}=U_{V(x,y)z}$;

(7) $T_{x,y}T_{y,x}=I-V_w+U_w$ for $w=x\circ y-U_xy^2$.

Note that Macdonald's Principle is applicable only to (1), (5), and (7).

Interpreting (1) in terms of operators in the homotope $\mathfrak{J}^{(y)}$ we see it reduces to $V_x^{(y)p}=V_{x^{p(y)}}^{(y)}$, thus to the relation $V_x^p=V_{x^p}$ in any algebra for which $p\mathfrak{J}=0$, and this in turn follows by identifying coefficients of λ^p in $\{\lambda^2I+\lambda V_x+U_x\}^p=U_{\lambda 1+x}^p=U_{(\lambda 1+x)^p}=U_{\lambda^p 1+x^p}$. Formula (1) was the basic result used by John Faulkner [1] to prove that the inner derivations of a Jordan algebra form a restricted Lie algebra in characteristic p.

Similarly (2) reduces to $V_x^{(y)}V_z^{(y)}=V_{x,z}^{(y)}+U_{x,z}^{(y)}$, which follows from the general formula $V_{x,z}=V_xV_z-U_{x,z}$.

(3) follows from "duality" in the universal multiplication envelope (i.e. application of the involution satisfying $U_x{}^*=U_x$; note that the "reverse" of $V_{x,y}=V_xV_y-U_{x,y}$ is $V_yV_x-U_{x,y}=V_{y,x}$). An alternate proof uses Koecher's Principle for the first time: it suffices to prove the identity when x (hence U_x) is invertible. In this case

$$\begin{aligned} V_{x,y}V_{x,z} &= V_{x,y}U_xU_x^{-1}V_{x,z} = U_xV_{y,x}V_{z,x}U_x^{-1} \\ &= U_x\{V_{y,U(x)z}+U_{y,z}U_x\}U_x^{-1} \qquad \text{(by (2))} \\ &= V_{U(x)y,z}+U_xU_{y,z} \qquad \text{(using linearized UQJ II).} \end{aligned}$$

For (4) we use the Koecher Principle to allow us to assume y is invertible. Then we can multiply on the right by U_y and obtain

$$V_x^{(y)}U_z^{(y)}+U_z^{(y)}V_x^{(y)}=U_{x\circ_y z,z}^{(y)} \qquad \text{(using UQJ III),}$$

where the relation $V_xU_z+U_zV_x=U_{x\circ z,z}$ is a linearization of UQJ II. (4) is the basic result which allows us to define the inner structure Lie algebra Instr $(\mathfrak{J})$ of a Jordan algebra $\mathfrak{J}$.

We use the Koecher Principle heavily in (5)–(6), because for invertible x and y we have

$$T_{x,y}=U_xU_{x^{-1}-y}=U_{y^{-1}-x}U_y.$$

Thus (6) is $U_xU_{x^{-1}-y}U_zU_{x^{-1}-y}U_x=U_w$ for $w=U_xU_{x^{-1}-y}z=T_{x,y}z$ by UQJ II. (7) is the special case of (6) when $z=1$ since $T_{x,y}1=1-x\circ y+U_xy^2=1-w$.

One approach to (5) is to note that for invertible y the operator

$T_{x,y}=U_{y^{-1}-x}U_y$ is just $U^{(y)}_{1^{(y)}-x}$ in the isotope $\mathfrak{J}^{(y)}$, so $T_{x,y}T_{-x,y} = T_{U(x)y,y}$ is just $U_{1-x}U_{1+x}=U_{1-x^2}$ in $\mathfrak{J}^{(y)}$, from which we obtain $T_{y,-x}T_{y,x}=T_{y,U(x)y}$ by duality (the reverse of $T_{x,y}$ is $T_{y,x}$). Clearly $T_{x,-y}=T_{-x,y}$ by inspection, so all four expressions agree.

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