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ANALYTICITY OF DETERMINANTS OF OPERATORS ON A BANACH SPACE

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ABSTRACT. If $F(z)$ is an analytic family of operators on a Banach space which is of finite rank for each z, then rank $F(z)$ is constant except for isolated points, and det $(I+F(z))$ and tr $F(z)$ are analytic. Similarly if $F(z)$ is meromorphic.

In this note, we consider an analytic family $F(z)$ of operators on an arbitrary complex Banach space $\mathfrak{X}$, such that the rank of $F(z)$ is finite for each z. We prove that the rank of $F(z)$ is constant on the domain of analyticity, except for isolated points, and that the trace of $F(z)$ and the determinant of $I+F(z)$ are analytic functions on that domain. This is not immediately clear, since the range of $F(z)$ need not lie in a fixed finite dimensional subspace which is independent of z. The proof uses the trace norm of Ruston [2]. We extend the result to a meromorphic family, and establish the standard formula for the logarithmic derivative of det $(I+F(z))$. For the definitions of trace and determinant, see [2] and [1, pp. 160–162].

1. THEOREM. *If $F(z)$ is analytic on a domain Ω and* rank $F(z)$ *is finite for each z, then there is an integer m such that* rank $F(z)=m$ *except at isolated points where* rank $F(z)<m$.

2. LEMMA. *If $F\in\mathfrak{B}(\mathfrak{X})$, then* rank $F\geqq N$ *iff there exist bounded projections P and Q of dimension N such that PFQ has rank N.*

PROOF. If rank $F<N$, then rank $PFQ\leqq$rank $F<N$. Conversely, if rank $F\geqq N$, there are $x_1, \cdots, x_N$ such that $Fx_1, \cdots, Fx_N$ are linearly independent. If P projects on the span of $Fx_1, \cdots, Fx_N$ and Q on the span of $x_1, \cdots, x_N$, then PFQ has rank N.

This lemma implies that *rank is upper semicontinuous*: if $F_n\rightarrow F$ in norm and rank $F_n\leqq m$, then rank $F\leqq m$. For if P and Q have the same dimension exceeding m, then det $PFQ=\lim$ det $PF_nQ=0$, where the determinants are with respect to fixed bases of $P\mathfrak{X}$ and $Q\mathfrak{X}$. (One may also conclude this by considering a determinant of the form det $[\langle x_i^*, Fx_j\rangle]$.)

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PROOF OF THEOREM 1. For each $k \geqq 0$, let $E_k = \{z \in \Omega \mid \text{rank } F(z) \leqq k\}$. Since $\Omega = \bigcup_{k=1}^{\infty} E_k$, E_k is uncountable for some integer k, and so there is a smallest integer m such that E_m has a point of accumulation within Ω. If P and Q are arbitrary projections with $\dim P = \dim Q > m$, then the determinant $d(z)$ of $PF(z)Q$, computed with respect to fixed bases of $P\mathfrak{X}$ and $Q\mathfrak{X}$, vanishes on E_m, and hence on all of Ω. Since P and Q are arbitrary, Lemma 2 implies that $E_m = \Omega$. Since m is minimal, E_{m-1} consists of isolated points.

This proof also shows that the rank of $F(z)$ is determined by its values on any set with an accumulation point in Ω, and hence that no analytic continuation of $F(z)$ can have rank exceeding m. The hypothesis of Theorem 1 can be weakened by assuming only that the set of points at which $F(z)$ has finite rank is uncountable; however, it does not suffice to assume only that $F(z)$ has finite rank on a set with an accumulation point in Ω, for if $F(z)$ is the infinite diagonal matrix

$$F(z) = \begin{pmatrix} a_1(z) & & & & & \\ & a_2(z) & & & & \\ & & \ddots & & & \\ & & & a_n(z) & & \\ & & & & \ddots & \end{pmatrix}$$

where $a_n(z) = (z-1)(z-1/2) \cdots (z-1/n)$, then $F(z)$ is analytic for $|z| < 1$, while rank $F(1/n) = n-1$.

If $F \in \mathfrak{B}(\mathfrak{X})$ has finite rank, then, following Ruston [2], [3] let $\beta(F)$ denote the operator norm of F and

$$\tau(F) = \inf \sum_{i=1}^{m} |x_i^*| \, |x_i|$$

where the infimum is taken over all representations $F = \sum_{i=1}^{m} \langle x_i^*, \cdot \rangle x_i$ of F. τ is a norm, and

(1) $|\text{tr } F| \leqq \tau(F)$,

(2) $\beta(F) \leqq \tau(F) \leqq \beta(F)$ rank F, and

(3) $\tau(AF) \leqq \beta(A)\tau(F)$ for any A in $\mathfrak{B}(\mathfrak{X})$.

3. THEOREM. *If $F(z)$ is analytic and the rank of $F(z)$ is finite for all z in Ω, then* tr $F(z)$ *is analytic, and* $d\,\text{tr}\,F(z)/dz = \text{tr } F'(z)$.

PROOF. By Theorem 1, rank $F(z) \leqq m < \infty$ for some integer m. The rank of $D(z, h) = h^{-1}[F(z+h) - F(z)]$ cannot exceed $2m$, so that rank $F'(z) \leqq 2m$ by upper semicontinuity of rank. Hence, by (1) and (2),

$$\left| h^{-1}[\operatorname{tr} F(z+h) - \operatorname{tr} F(z)] - \operatorname{tr} F'(z) \right| = \left| \operatorname{tr}(D(z,h) - F'(z)) \right|$$
$$\leqq \tau(D(z,h) - F'(z)) \leqq 4m\beta(D(z,h) - F'(z)).$$

But the final term tends to zero as $h \to 0$, since $F(z)$ is analytic in norm.

4. THEOREM. *If $F(z)$ is analytic and the rank of $F(z)$ is finite for all z in Ω, then $\Delta(z) = \det(I + F(z))$ is analytic.*

PROOF. Let z_0 be in Ω, and set $F_1(z) = F(z) - F(z_0)$. For z near z_0, $\beta(F_1(z)) < 1/2$, so that $I + F_1(z)$ is invertible and

$$I + F(z) = [I + F(z_0)(I + F_1(z))^{-1}][I + F_1(z)]. \tag{4}$$

The determinant of the first factor is analytic, since the range of $F(z_0)(I + F_1(z))^{-1}$ is contained in a fixed space [**1**, pp. 160–162] while [**1**, (5.60), p. 46]

$$\det(I + F_1(z)) = \exp\{\operatorname{tr} \log(I + F_1(z))\} \tag{5}$$

where

$$\log(I + F_1(z)) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (F_1(z))^k. \tag{6}$$

The right side of (6) converges uniformly in operator norm and contains a factor $F_1(z)$. Theorem 3 therefore applies to (6), and it follows that (5) is analytic.

5. COROLLARY. *If $F(z)$ is meromorphic and the rank of $F(z)$ is finite for all z in Ω, then $\Delta(z) = \det(I + F(z))$ and* tr $F(z)$ *are meromorphic.*

PROOF. Let z_0 be a pole of $F(z)$ of order p. Then $(z - z_0)^p F(z)$ is analytic, and hence tr $F(z) = (z - z_0)^{-p} \operatorname{tr}(z - z_0)^p F(z)$ is meromorphic, with the order of its pole not exceeding p. If we write $F(z) = P(z) + K(z)$ where $K(z_0) = 0$ and $P(z)$ is a polynomial in $(z - z_0)^{-1}$, then we have near z_0

$$\Delta(z) = \det(I + P(z)[I + K(z)]^{-1}) \det(I + K(z)).$$

The first factor is meromorphic since $P(z)$, being a polynomial, has its range contained in a fixed space; the second factor is analytic by Theorem 4.

6. THEOREM. *In Theorem* 4,

$$\Delta'(z)/\Delta(z) = \operatorname{tr}[I + F(z)]^{-1} F'(z) \tag{7}$$

provided that $\Delta(z)$ does not vanish identically.

PROOF. We first note that (7) applies to the first factor on the right side of (4), since the range of $F(z_0)(I+F_1(z))^{-1}$ is contained in a fixed finite dimensional space [1, pp. 248–249]. It also applies to the second factor; for since

$$\tau_1(F_1(z)^k) \leqq \tau(F_1(z))\beta(F_1(z))^{k-1} \leqq 2m\beta(F_1(z))^k \leqq (m/2^{k-1})$$

the series (6) converges uniformly in trace norm, and we have

$$\begin{aligned} \log \det(I + F_1(z)) &= \operatorname{tr} \log(I + F_1(z)) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \operatorname{tr} F_1(z)^k \end{aligned} \tag{8}$$

where the series is uniformly convergent. Taking the trace of

$$dF_1(z)^k/dz = F_1'F_1^{k-1} + F_1F_1'F^{k-2} + \cdots + F_1^{k-1}F_1'$$

(where F_1 and F_1' denote $F_1(z)$ and $F_1'(z)$ respectively) and using Theorem 3, we find that

$$d \operatorname{tr} F_1^k(z)/dz = k \operatorname{tr}(F_1(z)^{k-1}F_1'(z)).$$

Differentiating (8) term by term, we obtain for the logarithmic derivative of $\det(I+F_1(z))$

$$\sum_{k=1}^{\infty} (-1)^{k-1} \operatorname{tr}(F_1(z)^{k-1}F_1'(z)) = \operatorname{tr}\{(I + F_1(z))^{-1}F_1'(z)\}.$$

If we now take determinants in (4) and apply these results, we obtain

$$\begin{aligned} \Delta'/\Delta &= \operatorname{tr}\{F_1'(I + F_1)^{-1}\} \\ &\quad - \operatorname{tr}\{[I + F_0(I + F_1)^{-1}]^{-1}F_0(I + F_1)^{-1}F_1'(I + F_1)^{-1}\} \\ &= \operatorname{tr}\{[I + F_0(I + F_1)^{-1}]^{-1}F_1'(I + F_1)^{-1}\} \\ &= \operatorname{tr}\{[I + F_0 + F_1]^{-1}F_1'\} \\ &= \operatorname{tr}\{[I + F]^{-1}F'\} \end{aligned}$$

where $F_0 = F(z_0)$.

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