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DECOMPOSITIONS OF ABELIAN p-GROUPS

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ABSTRACT. Using some elementary properties of endomorphism rings and their radical ideals, an equivalence between the category of p-rings and the category of Boolean rings and some examples introduced by the author, it is shown that for every countable atomic Boolean algebra there is a p-group without elements of infinite height, standard basic subgroup and no proper isomorphic subgroups which contains a maximal lattice of summands isomorphic to the given Boolean algebra. Moreover, it is established that this lattice is representative in the sense that it determines, up to isomorphism, all the summands of the group.

THEOREM. *Let B be any countable atomic Boolean algebra. Then there exists a p-group G without elements of infinite height, no proper isomorphic subgroups, and with standard basic subgroup which contains a maximal lattice of summands isomorphic to B. Moreover, every summand of G is isomorphic to some element of the lattice.*

Let p be a prime number and $\mathcal{R}_p$ the category of p-rings with ring homomorphisms as morphisms. It is known that the functor which associates with each p-ring R its Boolean ring of idempotents $I(R)$ $(a\oplus b=a+b-2ab)$ and with each ring homomorphism $\gamma:R\to S$ its restriction $\gamma|I(R):I(R)\to I(S)$ is an equivalence between $\mathcal{R}_p$ and the category of Boolean rings $\mathcal{R}_2$ [**5**]. By a theorem of McCoy and Montgomery [**2**], it is known that each p-ring is isomorphic to a subdirect product of fields F_p (F_p denotes the prime field of characteristic p). In [**4**], it was shown that if R is a countable p-ring with identity such that $\oplus\sum_{i=1}^{\infty} F_p\subset R\subset \prod_{i=1}^{\infty} F_p$, then there exists a p-group $G=G(R)$ without elements of infinite height, no proper isomorphic subgroups and with a standard basic subgroup whose endomorphism ring $E(G)$ is isomorphic to R modulo its Jacobson radical $J(E(G))$. It is easily shown that the condition $\oplus\sum_{i=1}^{\infty} F_p+\{1\}\subseteq R \subset \prod_{i=1}^{\infty} F_p$ implies that $I(R)$ is an atomic Boolean algebra; and, moreover, if B is any countable atomic Boolean algebra, then there exists a countable p-ring $R(B)$ such that $\oplus\sum_{i=1}^{\infty} F_p+\{1\}\subseteq R(B)$

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$\subset \prod_{i=1}^{\infty} F_p$ and whose Boolean ring of idempotents is isomorphic to B. By Corollary 5.14 of [4] the p-group $G=G(R(B))$ has the property that $E(G)/J(E(G))\cong R(B)$. By 4.4 and 5.9 of [4], there is an additive group $\Gamma(R(B))$ of endomorphisms of G which maps onto $R(B)$. By the proof of 4.3 [4], $\Gamma(R(B))$ is a commutative subring of $E(G)$. It can be easily checked that $I(\Gamma(R(B)))$ is a Boolean ring of idempotents which maps onto $I(R(B))$. Since elements in a Boolean ring of idempotents contained within a ring are never identified modulo the radical of the ring, it follows that $I(\Gamma(R(B)))\cong I(R(B))\cong B$. The Boolean ring $I(\Gamma(R(B)))$ clearly splits G into a maximal lattice of summands isomorphic to B. If H is any summand of G and e is its associated idempotent, then, since $I(\Gamma(R(B)))$ maps onto $I(R(B))$, there exists an idempotent $f\in\Gamma(R(B))$ such that $e-f\in J(E(G))$. It follows that $1-(e-f)=\mu$ and $1-(f-e)=\nu$ are automorphisms of G. Moreover, $\mu(eG)\subseteq f(G)$ and $\nu(fG)\subseteq eG$. Hence, $\nu\mu|e(G):eG\rightarrow eG$ is an isomorphism of eG into eG. Since $G=e(G)\oplus(1-e)G$, it follows that $\nu\mu|eG$ must be onto, for otherwise G would be isomorphic to the proper subgroup $\nu\mu(eG)\oplus(1-e)G$. Consequently, μ maps eG onto fG and the two summands are isomorphic.

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