SEMILATTICE OF BISIMPLE REGULAR SEMIGROUPS

H. R. KRISHNA IYENGAR

ABSTRACT. The main purpose of this paper is to show that a regular semigroup S is a semilattice of bisimple semigroups if and only if it is a band of bisimple semigroups and that this holds if and only if $\mathfrak{D}$ is a congruence on S. It is also shown that a quasiregular semigroup S which is a rectangular band of bisimple semigroups is itself bisimple.

In [3, Theorem 4.4] it was shown that a semigroup S is a semi-lattice of simple semigroups if and only if it is a union of simple semigroups. The purpose of this paper is to obtain corresponding results for a semigroup which is a semilattice of bisimple regular semigroups. Unfortunately, a semilattice of bisimple semigroups need not be a union of bisimple semigroups as illustrated by a simple ω-semigroup constructed by Munn [5]. However, we get some equivalent conditions for such semigroups. In particular we show that a regular semigroup is a semilattice of bisimple semigroups if and only if it is a band of bisimple semigroups.

1. **Equivalent conditions.** In this section we consider a set of equivalent conditions for a semigroup S to be a semilattice of bisimple semigroups. We adopt the terminology and notation of [2].

LEMMA 1.1. *Let S be a semilattice Ω of semigroups S_α and let D be a $\mathfrak{D}$-class of S. Then, either $S_\alpha \cap D = \square$ or $D \subseteq S_\alpha$.*

PROOF. Suppose $S_\alpha \cap D \neq \square$. Let $a_\alpha \in S_\alpha \cap D$. If $b_\beta \in S_\beta$ and $a_\alpha \mathfrak{D} b_\beta$, then there exists $c_\gamma \in S_\gamma$ $(\gamma \in \Omega)$ such that $a_\alpha \mathfrak{R} c_\gamma$ and $c_\gamma \mathfrak{L} b_\beta$. Also $a_\alpha \mathfrak{R} c_\gamma$ implies that either $a_\alpha = c_\gamma$ in which case $\gamma = \alpha$, or there exist $x_\lambda \in S_\lambda$, $y_\mu \in S_\mu$ $(\lambda, \mu$ in $\Omega)$ such that $a_\alpha x_\lambda = c_\gamma$ and $c_\gamma y_\mu = a_\alpha$. However, since S is a semilattice of the semigroups S_α, $a_\alpha x_\lambda \in S_{\alpha\lambda}$ and $c_\gamma y_\mu \in S_{\gamma\mu}$. It follows that $\gamma = \alpha\lambda$ and $\alpha = \gamma\mu$ and so $\gamma \leq \alpha$ and $\alpha \leq \gamma$. Thus, in either case, $\gamma = \alpha$. Likewise, $\gamma = \beta$. Therefore, $\alpha = \beta$ and $D \subseteq S_\alpha$.

LEMMA 1.2. *Let S be a semigroup. If $ab\mathfrak{D}ba$ for all a, b in S, then $\mathfrak{D}$ is a congruence on S.*

PROOF. Let $a\mathfrak{D}b$. Then there exists $x \in S$ such that $a\mathfrak{L}x$ and $x\mathfrak{R}b$. Since $\mathfrak{L}$ is a right congruence and $\mathfrak{R}$ is a left congruence, we have

for $c \in S$, $ac \mathcal{L} xc$ and $cx \mathcal{R} cb$. That is, $ac \mathfrak{D} xc$ and $cx \mathfrak{D} cb$. Since by hypothesis $xc \mathfrak{D} cx$, and $cb \mathfrak{D} bc$, then $ac \mathfrak{D} bc$ and also $ca \mathfrak{D} cb$. Thus $\mathfrak{D}$ is a congruence on S.

LEMMA 1.3. *Let S be a semigroup which is a semilattice of bisimple semigroups. Then $\mathfrak{D}$ is a congruence on S.*

PROOF. Let $S = \bigcup_{\alpha \in \Omega} S_\alpha$ be a semilattice of the bisimple semigroups S_α. Then, by Lemma 1.1 each S_α is a union of $\mathfrak{D}$-classes of S. However, since S_α is bisimple, any two elements of S_α are $\mathfrak{D}$-related in S_α and hence in S. Consequently, S_α consists of a single $\mathfrak{D}$-class of S. Thus the $\mathfrak{D}$-classes of S are just the subsemigroups S_α and hence $\mathfrak{D}$ is a congruence on S.

As defined in [4], an element x in a semigroup S is said to be *quasiregular* if there exist elements a, b, c, $d \in S^1$ such that $x = xaxb = cxdx$. A semigroup S is *quasiregular* if every element of S is quasiregular. From the results in [4], we find that a regular semigroup is quasiregular, but the converse does not necessarily hold.

THEOREM 1.4. *A quasiregular semigroup S, which is a rectangular band of bisimple semigroups, is itself bisimple.*

PROOF. Let $S = \bigcup \{ S_{i\lambda} \mid i \in I, \lambda \in \Lambda \}$ be a rectangular band of the bisimple semigroups $S_{i\lambda}$ and let S be quasiregular. Let $x_{i\lambda} \in S_{i\lambda}$, $y_{j\mu} \in S_{j\mu}$. Since $x_{i\lambda} y_{j\mu}$ is quasiregular, there exist elements a, b, c, d in S^1 such that

$$(1.4.1) \qquad\qquad x_{i\lambda} y_{j\mu} a x_{i\lambda} y_{j\mu} b = x_{i\lambda} y_{j\mu}$$

and

$$(1.4.2) \qquad\qquad c x_{i\lambda} y_{j\mu} d x_{i\lambda} y_{j\mu} = x_{i\lambda} y_{j\mu}.$$

However, (1.4.1) implies that $x_{i\lambda} y_{j\mu} a x_{i\lambda} \mathfrak{R} x_{i\lambda} y_{j\mu}$ and $(1 \cdot 4 \cdot 2)$ implies that $x_{i\lambda} y_{j\mu} \mathcal{L} y_{j\mu} d x_{i\lambda} y_{j\mu}$, and hence $x_{i\lambda} y_{j\mu} a x_{i\lambda} \mathfrak{D} y_{j\mu} d x_{i\lambda} y_{j\mu}$. Since $x_{i\lambda} y_{j\mu} a x_{i\lambda} \in S_{i\lambda}$ and $y_{j\mu} d x_{i\lambda} y_{j\mu} \in S_{j\mu}$, and $S_{i\lambda}$ and $S_{j\mu}$ are bisimple, it follows that $S_{i\lambda}$ and $S_{j\mu}$ are contained in a single $\mathfrak{D}$-class of S for all $i, j \in I$ and $\lambda, \mu \in \Lambda$. Thus S is bisimple.

THEOREM 1.5. *The following are equivalent for any regular semigroup S.*

(A) *S is a semilattice of bisimple semigroups.*

(B) *For a, $b \in S$, $ab \mathfrak{D} ba$.*

(C) *$\mathfrak{D}$ is a congruence on S.*

(D) *If e, f, $g \in E(S)$, where $E(S)$ is the set of idempotents of S, then $e \mathfrak{D} f \Rightarrow$ (i) $eg \mathfrak{D} fg$ and (ii) $ge \mathfrak{D} gf$.*

(E) *S is a band of bisimple semigroups.*

Proof. Assume (A) and let S be a semilattice Ω of bisimple semigroups S_α. Let a, $b \in S$, and denote the $\mathfrak{D}$-classes containing a and b by D_a and D_b respectively. Then, by Lemma 1.1, $D_a \subseteq S_\alpha$ and $D_b \subseteq S_\beta$ for some α, β in Ω. Hence $ab \in D_a D_b \subseteq S_\alpha S_\beta \subseteq S_{\alpha\beta}$ and $ba \in D_b D_a \subseteq S_\beta S_\alpha \subseteq S_{\alpha\beta}$. Since $S_{\alpha\beta}$ is bisimple, we conclude that $ab \mathfrak{D} ba$. Thus (A) implies (B). (B) implies (C) by Lemma 1.2. Also (C) implies (D) trivially. We shall now show that (D) implies (C). Suppose $a \mathfrak{D} b$ and $c \in S$. Let a', b', c' be one of the inverses of a, b, c respectively. Since $a'a \mathfrak{L} a$ and $\mathfrak{L}$ is a right congruence, we have $a'a \cdot cc' \mathfrak{L} a \cdot cc'$. Also, since $cc' \mathfrak{R} c$ and since $\mathfrak{R}$ is a left congruence, we have $a \cdot cc' \mathfrak{R} ac$. Thus $a'acc' \mathfrak{D} ac$ and likewise $b'bcc' \mathfrak{D} bc$. Since $a'a \mathfrak{D} b'b$, it follows from (D) that $a'acc' \mathfrak{D} b'bcc'$ and hence $ac \mathfrak{D} bc$. In a similar manner, $ca \mathfrak{D} cb$ and thus $\mathfrak{D}$ is a congruence.

To show that (C) implies (E), suppose $a \mathfrak{D} b$ so that $a \mathfrak{D} bb'$ where b' is an inverse of b. Since $\mathfrak{D}$ is a congruence, this implies that $ab \mathfrak{D} bb'b$ or $ab \mathfrak{D} b$. Thus each $\mathfrak{D}$-class of S is a subsemigroup of S and moreover, since S is regular, it is a bisimple semigroup [2, p. 61, Example 6]. Denoting the $\mathfrak{D}$-classes of S by S_α, $\alpha \in \Omega$, we have $S = \cup \{ S_\alpha \mid \alpha \in \Omega \}$. Further, since $\mathfrak{D}$ is a congruence, $S_\alpha S_\beta$ is contained in a semigroup S_γ for some $\gamma \in \Omega$. If we define $\alpha\beta = \gamma$, then, noting that $S_\alpha S_\alpha \subseteq S_\alpha$, Ω becomes a band and thus S is a band of bisimple semigroups.

Finally, in order to show that (E) implies (A), we use the following result by Clifford [1]. "Let $\mathfrak{C}$ be a class of semigroups. If a semigroup S is a band of semigroups of type $\mathfrak{C}$, then S is a semilattice of semigroups each of which is a rectangular band of semigroups of type $\mathfrak{C}$." Now assuming that S is a band of bisimple semigroups, S is a semilattice Ω of semigroups S_α $(\alpha \in \Omega)$, where each S_α is a rectangular band of bisimple semigroups. If $a_\alpha \in S_\alpha \subseteq S$, since S is regular, there exists $a'_\alpha \in S$ such that $a_\alpha a'_\alpha a_\alpha = a_\alpha$ and further $a_\alpha \mathfrak{D} a'_\alpha$. However, by Lemma 1.1 each S_α is a union of $\mathfrak{D}$-classes of S and hence $a'_\alpha \in S_\alpha$. Thus S_α is regular, and in particular, quasiregular. Consequently, by Theorem 1.4, S_α is bisimple, and we have (A). This completes the theorem.

2. **Examples.** In this section we consider some examples to show that Theorem 1.5 is not true in general for semigroups which are not regular. First we need a lemma.

Lemma 2.1. *Let S be a Baer-Levi semigroup* [3, p. 82] *and T be any semigroup. Let $A = T \times S$ be the direct product of T and S. Then (t, s) $\mathfrak{D}(t', s')$ if and only if $t \mathfrak{R} t'$, in T, and consequently the $\mathfrak{D}$-classes of A are just the sets $R \times S$ where R is an $\mathfrak{R}$-class of T.*

PROOF. The proof of the "only if" part follows if we show that the $\mathcal{L}$-classes of A contain single elements. Now suppose $(t, s)\mathcal{L}(t', s')$. Then either

(i) $(t, s) = (t', s')$, or

(ii) there exists x, y in T and a, b in S such that $(x, a)(t, s) = (t', s')$ and $(y, b)(t', s') = (t, s)$.

If (ii) holds, we have $as = s'$ and $bs' = s$. Thus, $(ba)s = b(as) = bs' = s$ which contradicts [3, Lemma 8.3]. Hence (ii) cannot hold, and we have established the assertion.

The "if" part is clear, since S is right simple and any two elements of S are $\mathcal{R}$-related.

EXAMPLE 1. Let $E_1 = \{e_1, e_2\}$ be a right zero semigroup of order 2, $E_2 = \{e_3, e_4, e_5\}$ be a left zero semigroup of order 3, and let $E = E_1 \cup E_2$ be a disjoint union where

$$e_i e_j = e_j e_i = e_j \quad \text{for } i = 1, 2, \ j = 3, 4,$$

$$e_1 e_5 = e_3, \qquad e_5 e_1 = e_5, \qquad e_2 e_5 = e_4, \qquad e_5 e_2 = e_5.$$

The fact that E is a semigroup can be easily verified. Moreover, $e_1 \mathcal{R} e_2$ and $e_3 \mathcal{L} e_4 \mathcal{L} e_5$. Set $T = E \times S$, the direct product of E and S, where S is the Baer-Levi semigroup. Using Lemma 2.1, the $\mathfrak{D}$-classes of T are given by

$$D_1 = \{e_1, e_2\} \times S, \qquad D_i = \{e_{i+1}\} \times S \ (i = 2, 3, 4).$$

Clearly, T is a band of the bisimple semigroups $S_i = \{e_i\} \times S$, $i = 1$, 2, 3, 4, 5. However, if $s \in S$, then $(e_1, s)\mathfrak{D}(e_2, s)$. But $(e_1, s)(e_5, s)$ $= (e_3, s^2) \in D_2$ and $(e_2, s)(e_5, s) = (e_4, s^2) \in D_3$. Thus $\mathfrak{D}$ is not a congruence. Moreover, T is not a semilattice of bisimple semigroups, for otherwise, Lemma 1.3 would imply that $\mathfrak{D}$ is a congruence. Further T does not satisfy the condition (B) of Theorem 1.5, since in that case Lemma 1.2 would again imply that $\mathfrak{D}$ is a congruence.

EXAMPLE 2. Let $S = \{0, a, 1\}$ where $a^2 = 0$, with 0 as the zero element and 1 as the identity element. Then $\mathfrak{D}$ is a congruence, being the identity relation. However, S is neither a band of bisimple semigroups nor a semilattice of bisimple semigroups.

EXAMPLE 3. Let $E = \{e_1, e_2\}$ be a left zero semigroup and let S be the Baer-Levi semigroup. Set $T = E \times S$. Then T is a semigroup and by Lemma 2.1, the $\mathfrak{D}$-classes of T are:

$$D_1 = \{e_1\} \times S \quad \text{and} \quad D_2 = \{e_2\} \times S.$$

Moreover, D_1 and D_2 are bisimple semigroups such that $D_1 D_2 \subseteq D_1$ and $D_2 D_1 \subseteq D_2$. Thus $\mathfrak{D}$ is a congruence on T and T is a band of bi-

simple semigroups. However, if $a = (e_1, s)$ and $b = (e_2, s)$ for $s \in S$, then $ab \in D_1$ and $ba \in D_2$. Hence, ab and ba are not $\mathfrak{D}$-related. This will further imply that T is not a semilattice of bisimple semigroups.

REMARK. If, in Example 3 above, we write $S_{11} = D_1$, $S_{21} = D_2$, we find that S is a rectangular band of the bisimple semigroups S_{11} and S_{21}. But S is not bisimple. This shows that Theorem 1.4 is not true in general for semigroups which are not quasiregular.

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THE UNIVERSITY OF WISCONSIN-GREEN BAY, GREEN BAY, WISCONSIN 54301