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## SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

# SHORT PROOF OF SOBCZYK'S THEOREM

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ABSTRACT. A new proof is given of Sobczyk's Theorem, which asserts that $c_0$ is complemented in any separable Banach space which contains it (isomorphically) as a closed subspace.

Sobczyk's Theorem says that for every separable Banach space $X$ and isometric copy $Y$ of $c_0$ contained in $X$, there is a projection of norm at most 2 from $X$ to $Y$ (and 2 is best possible). This is a consequence of setting $T_0$=identity in the following theorem, whose equivalence to Sobczyk's Theorem seems to be part of the folklore.

THEOREM. *Let $X$ be a separable Banach space and $Y$ a closed subspace of $X$. If $T_0: Y \to c_0$ is a linear operator of norm $\lambda$, there exists an extension $T: X \to c_0$ of norm at most $2\lambda$.*

PROOF. $S_\lambda$ denotes the closed ball of radius $\lambda$ in the dual of $X$. In the weak * topology $S_\lambda$ is compact, and because $X$ is separable, metrizable. Let $d(\cdot, \cdot)$ be a compatible metric. By the Hahn-Banach theorem there exists a sequence $\phi_1, \phi_2, \cdots$ in $S_\lambda$ such that the $n$th coordinate of $T_0 y$, $y \in Y$, is $(T_0 y)_n = \phi_n(y)$. Letting $K = S_\lambda \cap Y^\perp$, it is clear every cluster point of $\{\phi_n\}$ lies in $K$. What is the same, $d(\phi_n, K) \to 0$, and if $\{\psi_n\} \subseteq K$ is a sequence minimizing $d(\phi_n, \psi)$, $\psi \in K$, 0 is the only cluster point of $\{\phi_n - \psi_n\} \subseteq S_{2\lambda}$. Define $(Tx)_n = \phi_n(x) - \psi_n(x)$,[2] $x \in X$. $T$ maps $X$ to $c_0$, has norm at most $2\lambda$, and agrees with $T_0$ on $Y$. The proof is complete.

Haskell Rosenthal has pointed out to us that our argument uses only the weak * metrizability of the closure of $\{\phi_n\}$ in $S_\lambda$ and there-

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[2] ADDED IN PROOF. Since this was written we have learned this device has been used earlier by Köthe for another result, also equivalent to Sobczyk's Theorem [3], [4].

fore is valid for any "weakly compactly generated" Banach space $X$ (those Banach spaces $X$ having weakly compact total subsets).

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