

SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

SHORT PROOF OF SOBCZYK'S THEOREM

WILLIAM A. VEECH¹

ABSTRACT. A new proof is given of Sobczyk's Theorem, which asserts that c_0 is complemented in any separable Banach space which contains it (isomorphically) as a closed subspace.

Sobczyk's Theorem says that for every separable Banach space X and isometric copy Y of c_0 contained in X , there is a projection of norm at most 2 from X to Y (and 2 is best possible). This is a consequence of setting $T_0 = \text{identity}$ in the following theorem, whose equivalence to Sobczyk's Theorem seems to be part of the folklore.

THEOREM. *Let X be a separable Banach space and Y a closed subspace of X . If $T_0: Y \rightarrow c_0$ is a linear operator of norm λ , there exists an extension $T: X \rightarrow c_0$ of norm at most 2λ .*

PROOF. S_λ denotes the closed ball of radius λ in the dual of X . In the weak* topology S_λ is compact, and because X is separable, metrizable. Let $d(\cdot, \cdot)$ be a compatible metric. By the Hahn-Banach theorem there exists a sequence ϕ_1, ϕ_2, \dots in S_λ such that the n th coordinate of $T_0 y$, $y \in Y$, is $(T_0 y)_n = \phi_n(y)$. Letting $K = S_\lambda \cap Y^\perp$, it is clear every cluster point of $\{\phi_n\}$ lies in K . What is the same, $d(\phi_n, K) \rightarrow 0$, and if $\{\psi_n\} \subseteq K$ is a sequence minimizing $d(\phi_n, \psi)$, $\psi \in K$, 0 is the only cluster point of $\{\phi_n - \psi_n\} \subseteq S_{2\lambda}$. Define $(Tx)_n = \phi_n(x) - \psi_n(x)$,² $x \in X$. T maps X to c_0 , has norm at most 2λ , and agrees with T_0 on Y . The proof is complete.

Haskell Rosenthal has pointed out to us that our argument uses only the weak* metrizable of the closure of $\{\phi_n\}$ in S_λ and there-

Received by the editors July 30, 1970.

AMS 1970 subject classifications. Primary 46B99.

Key words and phrases. Separable Banach space, projection, weakly compactly generated Banach space.

¹ Research supported by NSF grant GP-18961.

² ADDED IN PROOF. Since this was written we have learned this device has been used earlier by Köthe for another result, also equivalent to Sobczyk's Theorem [3], [4].

Copyright © 1971, American Mathematical Society

fore is valid for any “weakly compactly generated” Banach space X (those Banach spaces X having weakly compact total subsets).

REFERENCES

1. N. Dunford and J. T. Schwartz, *Linear operators. I: General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR 22 #8302.
2. A. Sobczyk, *Projection of the space (m) on its subspace (c_0)* , Bull. Amer. Math. Soc. 47 (1941), 938–947. MR 3, 205.
3. G. Köthe, *Über einen Satz von Sobczyk*, An. Fac. Ci. Univ. Porto 49 (1966), 281–286. MR 37 #3322.
4. S. Goldberg, *On Sobczyk's projection theorem*, Amer. Math. Monthly 76 (1969), 523–526. MR 39 #6054.

RICE UNIVERSITY, HOUSTON, TEXAS 77001