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THE NORM OF A HERMITIAN ELEMENT IN A BANACH ALGEBRA

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ABSTRACT. We prove that the norm of a hermitian element in a Banach algebra is equal to the spectral radius of the element.

An element h in a complex Banach algebra with identity (of norm 1) is said to be *hermitian* if $\|\exp i\alpha h\| = 1$ for all real α [6], [3, Definition 5.1]. I. Vidav uses a Phragmén-Lindelöf theorem to show that the numerical radius [3, Definition 2.1] of a hermitian element is equal to its spectral radius [6, p. 123, Hilfssatz 3], [3, Theorem 5.10]. We show that the norm of $h+\beta 1$ is equal to the spectral radius of $h+\beta 1$ for h a hermitian element and β a complex number (Proposition 2). The proof uses a generalisation of Bernstein's theorem which gives a bound on the derivative of an entire function along the real line. F. F. Bonsall and M. J. Crabb [2] have recently given an elementary proof of our Proposition 2 when β is zero (which is equivalent to β real). In Lemma 5 and Proposition 6 we construct a norm on the algebra of polynomials, in one indeterminate x, which is maximal with respect to the property that x is hermitian of norm one.

An entire function F is said to be of *order R* if

$$R = \limsup_{\alpha\to\infty} \frac{\log\log M(\alpha)}{\log \alpha}$$

where $M(\alpha)$ denotes $\sup\{|F(z)| : |z| \leqq \alpha\}$. An entire function of finite order R is said to be of *type T* if

$$T = \limsup_{\alpha\to\infty} \alpha^{-R}\log M(\alpha).$$

If the entire function F is of order less than 1 or F is of order 1 and type less than or equal to T, we say F is of *exponential type T* [1, p. 8]. G. Lumer and R. S. Phillips [5, p. 685, Theorem 2.3] prove the following lemma when x is topologically nilpotent. Let $\nu(x)$ denote the spectral radius of an element x.

1. LEMMA. *Let A be a Banach algebra with identity. For each continuous linear functional f on A and each x in A, the entire function* $\lambda \to f(\exp \lambda x)$ *is of exponential type* $\nu(x)$.

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PROOF. Since $|f(\exp \lambda x)| \leqq \|f\| \cdot \exp |\lambda| \cdot \|x\|$, we see that the order of $f(\exp \lambda x)$ is less than or equal to 1. Suppose that the order of $f(\exp \lambda x)$ is 1. The nth derivative of $f(\exp \lambda x)$ at zero is $f(x^n)$. Thus, by equation 2.2.12 of [1, p. 11], the type of $f(\exp \lambda x)$ is equal to $\limsup_{n\to\infty} |f(x^n)|^{1/n}$, which is less than or equal to the spectral radius of x. This completes the proof.

Alternatively Lemma 1 may be proved using [3, Theorem 3.8].

For each x in A that is not topologically nilpotent there is a continuous linear functional f on A with $\|f\| = f(1) = 1$ such that $f(\exp \lambda x)$ has order 1 and type $\nu(x)$. Let B be a closed commutative subalgebra of A containing x and 1, and let θ be a character on B such that the modulus of $\theta(x)$ is equal to $\nu(x)$. By the Hahn-Banach theorem there is an extension f of θ to A of norm 1. Then $f(\exp \lambda x) = \exp \lambda\, \theta(x)$, which is of order 1 and type $\nu(x)$.

2. PROPOSITION. *Let A be a Banach algebra with identity. Then $\|h+\beta 1\| = \nu(h+\beta 1)$ for each hermitian element h and each complex number β.*

PROOF. Because the sum of two hermitian elements is hermitian and a real multiple of the identity is hermitian [6, p. 122, Hilfssatz 2], [3, Lemma 5.4], we have to prove $\|h+\beta 1\| = \nu(h+\beta 1)$ only when β is imaginary. Let γ be a real number, and let f be a continuous linear functional on A of norm 1 with $f(h+i\gamma 1) = \|h+i\gamma 1\|$. Then, by Lemma 1, $\lambda \to f(\exp \lambda i h)$ is an entire function of exponential type $\nu(h)$ whose modulus is bounded by 1 for all real λ. We now state a generalization of a theorem of S. Bernstein [4, Theorem 1], [1, Chapter 11]. If F is an entire function of exponential type T whose modulus is bounded by 1 for all real λ, then

$$|F'(\lambda) - \alpha F(\lambda)| \leqq (T^2 + \alpha^2)^{1/2} \tag{1}$$

for all real λ and α, where $'$ denotes differentiation with respect to λ. Although the hypotheses of [4, Theorem 1] are not stated in terms of the type of an entire function it is a routine matter to write them in this form so that (1) is a special case of [4, Theorem 1]. Alternatively, when T is nonzero this inequality may be obtained from inequality 11.4.5 of [1, p. 214] by substituting $\alpha = -T \tan \omega$ (see also [1, p. 211 and p. 222]).

We apply (1) with $F(\lambda) = f(\exp \lambda i h)$ and $\lambda = 0$ obtaining

$$\|h + i\gamma 1\| = |f(h) + i\gamma f(1)| \leqq |\nu(h) + i\gamma| \tag{2}$$

since the derivative of $f(\exp \lambda i h)$ is $f(ih \exp \lambda i h)$. Since the spectrum of h is contained in the real line [6, p. 122, Hilfssatz 2] and γ is real,

$$\nu(h + i\gamma 1) = |\nu(h) + i\lambda| . \tag{3}$$

Combining (2) and (3) completes the proof.

We shall require the following corollary in Proposition 6.

3. Corollary. *If Q is a polynomial, with complex coefficients, whose zeros lie on the imaginary axis, and if h is a hermitian element, then* $\|Q(h)\| = |Q(\|h\|)|$.

Proof. The spectrum of h is contained in the real line, and so, by Proposition 2, $\|h\|$ or $-\|h\|$ is in $\sigma(h)$. Thus $\nu(h-\alpha 1) = | \|h\| - \alpha|$ for all imaginary α. Proposition 2 now implies that $\|h-\alpha 1\| = | \|h\| - \alpha|$. We factorise $Q(h)$ into linear factors and use this result and the submultiplicativity of the norm to obtain $\|Q(h)\| \leqq |Q(\|h\|)|$. As all the zeros of Q lie on the imaginary axis, $|Q(\|h\|)| = |Q(-\|h\|)|$. This and the result that $\|h\|$ or $-\|h\|$ is in $\sigma(h)$ imply that $|Q(\|h\|)| \leqq \nu(Q(h)) \leqq \|Q(h)\|$, which completes the proof.

Alternatively Corollary 3 may be proved directly from Lemma 1 by using Theorems 11.7.7, 7.8.3, and 11.7.2 of [1].

4. Definition. Let $\mathbf{C}(x)$ be the algebra of all polynomials in x with complex coefficients, and let L be the set of all constants, and all polynomials whose zeros lie on the imaginary axis in the complex plane. Then every polynomial P in $\mathbf{C}(x)$ is the sum of a finite number of polynomials in L. Let α be positive real number. We define $\|\cdot\|_0$ on $\mathbf{C}(x)$ by

$$\|P\|_0 = \inf\left\{\sum_j |Q_j(\alpha)| : P = \sum_j Q_j,\ Q_j \in L \text{ all } j\right\},$$

and $\|\cdot\|_\infty$ on $\mathbf{C}(x)$ by

$$\|P\|_\infty = \sup\{|P(\lambda)| : -\alpha \leqq \lambda \leqq \alpha\}.$$

5. Lemma. *Let α be a positive real number. Then $\|\cdot\|_0$ (and $\|\cdot\|_\infty$) is an algebra norm on $\mathbf{C}(x)$, $\|\cdot\|_0 \geqq \|\cdot\|_\infty$, and x is a hermitian element in the completion of $(\mathbf{C}(x), \|\cdot\|_0)$ with spectrum the interval $[-\alpha, \alpha]$.*

Proof. If Q is in L, then $\beta \to |Q(\beta)|$ is a monotonically increasing function of positive real β, as may be seen by factorising Q into linear factors and noting that the zeros of Q lie on the imaginary axis so that $\beta \to |\beta - \gamma|$ is a monotonically increasing function for each zero γ of Q. Let $P = \sum_j Q_j$ with Q_j in L, and let $-\alpha \leqq \lambda \leqq \alpha$. Then $|P(\lambda)| \leqq \sum_j |Q_j(\lambda)| = \sum_j |Q_j(|\lambda|)|$ since λ is real, since the zeros of Q_j lie on the imaginary axis, and since $|\lambda + i\alpha| = |-\lambda + i\alpha|$ for all α. Therefore $|P(\lambda)| \leqq \sum_j |Q_j(\alpha)|$, so that $\|P\|_\infty \leqq \|P\|_0$. If $\|P\|_0 = 0$, P is zero on $[-\alpha, \alpha]$ and so $P=0$. An elementary calculation now shows that $\|\cdot\|_0$ is an algebra norm on $\mathbf{C}(x)$.

Let A be the completion of $\mathbf{C}(x)$ in $\|\cdot\|_0$. Then, for all real t, exp itx is the $\|\cdot\|_0$=limit of $(1+i/n\cdot tx)^n$ as n tends to infinity [3, Theorem 3.3]. Now $\|(1+i/n\cdot tx)^n\|_0 \leqq |(1+i/n\cdot t\alpha)^n|$, so that, taking limits as n tends to infinity, we obtain $\|\exp itx\|_0 \leqq |\exp it\alpha| = 1$. Therefore $\|\exp itx\|_0 = 1$ for all real t, so that x is a hermitian element in A.

Since x is hermitian and $\|x\|_0 \leqq \alpha$, the spectrum of x in A is contained in the interval $[-\alpha, \alpha]$. For each λ in $[-\alpha, \alpha]$ the function $P \to P(\lambda)$: $\mathbf{C}(x) \to \mathbf{C}$ is a continuous character on $\mathbf{C}(x)$ taking the value λ at x. This shows that the spectrum of x in A is $[-\alpha, \alpha]$, and completes the proof.

The norm $\|\cdot\|_0$ given above is the maximal norm on $\mathbf{C}(x)$ such that x is hermitian with $\|x\| = \alpha$.

6. PROPOSITION. *Let α be a positive real number, let A be a Banach algebra with identity, and let h be in A. Then h is hermitian with $\|h\| \leqq \alpha$ if, and only if, $\|P(h)\| \leqq \|P\|_0$ for all P in $\mathbf{C}(x)$.*

PROOF. If h is hermitian with $\|h\| \leqq \alpha$, then, for each Q in L, $\|Q(h)\| = |Q(\|h\|)| \leqq |Q(\alpha)|$ by Corollary 3 and the monotonicity of $|Q(\alpha)|$, which we proved in Lemma 5. Thus $\|P(h)\| \leqq \sum_j |Q_j(\alpha)|$ for all Q_j in L with $P = \sum_j Q_j$, so that $\|P(h)\| \leqq \|P\|_0$ for all P in $\mathbf{C}(x)$.

Conversely, suppose that $\|P(h)\| \leqq \|P\|_0$ for all P in $\mathbf{C}(x)$. Then, by [3, Theorem 3.3],

$$\|\exp ith\| = \lim_{n\to\infty} \|(1 + i/n\cdot th)^n\| \leqq \lim_{n\to\infty} \|(1 + i/n\cdot tx)^n\|_0$$
$$= \|\exp itx\|_0 = 1$$

for all real t. This implies that $\|\exp ith\| = 1$ for all real t, and completes the proof since $\|h\| \leqq \|x\|_0 \leqq \alpha$.

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