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EMBEDDING OF COMPLETE MOORE SPACES

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ABSTRACT. The purpose of this paper is to give necessary and sufficient conditions for Moore spaces to be complete in terms of embeddings in bicompact spaces.

In [2], Čech showed that a metric space X is complete if and only if there exists a bicompact Hausdorff space $Y \supset X$ such that X is G_δ in Y. We shall show that this result can be extended to completely regular Moore spaces. It is also shown that the condition of complete regularity can be eliminated by replacing Hausdorff with T_1 and placing additional conditions on the embedding of X in Y.

Some of the results of this paper were announced in [3].

Terms which are not defined within are used as in [4].

DEFINITION 1. A sequence $\{\mathcal{G}_i\}_{i=1}^{\infty}$ of open covers of a topological space X is a *development* for X if (1) $\mathcal{G}_{i+1}$ is a refinement for $\mathcal{G}_i$, and (2) if x is a point of X and U is an open set in X containing x, then there is a natural number k such that $\mathrm{St}(x, \mathcal{G}_k) \subset U$. A *Moore space* is a regular T_1-space which has a development. A Moore space is *complete with respect to* the development $\{\mathcal{G}_n\}_{n=1}^{\infty}$ provided that, if $\{A_i\}_{i=1}^{\infty}$ is a nonincreasing sequence of nonempty closed sets such that, for each n, there is a $G_n \in \mathcal{G}_n$ with $A_n \subset G_n$, then $\bigcap_{i=1}^{\infty} A_i \neq \emptyset$. A Moore space is *complete* if it has a development with respect to which it is complete. See [5].

DEFINITION 2. A subspace X of a topological space Y is *regularly embedded* in Y provided that if p is a point of X and U is an open subset of Y containing p, then there is an open subset V of Y containing p whose closure is contained in U.

It should be noticed here that any subspace of a regular space is regularly embedded in it and, if X is regularly embedded in Y, then X is regular. However, neither of these two implications can be reversed.

THEOREM 3. *Let X be a Moore space. If there is a bicompact space Y*

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in which X is regularly embedded such that X is a G_δ subset of Y, then X is complete.

PROOF. Let $\{\mathcal{G}_i\}_{i=1}^\infty$ be a development for X and let $\{P_i\}_{i=1}^\infty$ be a sequence of open subsets of Y such that $X=\bigcap_{i=1}^\infty P_i$. For each natural number i and each point x in X, let $G_i(x)$ be an open subset of Y containing x such that $G_i(x)\cap X\in\mathcal{G}_i$. For each $x\in X$, let $\{H_i(x)\}_{i=1}^\infty$ be a sequence of open subsets of Y each of which contains x such that, for each i, $\mathrm{Cl}_Y\, H_i(x)\subset G_i(x)\cap P_i$ and $H_{i+1}(x)\subset H_i(x)$. Let $\mathcal{H}_i=\{H_i(x)\cap X : x\in X\}$. Since $\mathcal{H}_{i+1}$ refines $\mathcal{H}_i$ and $\mathcal{H}_i$ refines $\mathcal{G}_i$, the sequence $\{\mathcal{H}_i\}_{i=1}^\infty$ is a development for X. Let $\{A_i\}_{i=1}^\infty$ be a nonincreasing sequence of nonempty closed subsets of X such that, for each i, there is a $H_i'\in\mathcal{H}_i$ such that $A_i\subset H_i'$. The sequence $\{\mathrm{Cl}_Y\, A_i\}_{i=1}^\infty$ is a nonincreasing sequence of nonempty bicompact sets and, hence, $\bigcap_{i=1}^\infty \mathrm{Cl}_Y\, A_i\neq\emptyset$. But, for each i, $\mathrm{Cl}_Y\, A_i\subset\mathrm{Cl}_Y\, H_i'\subset P_i$. Thus, $\bigcap_{i=1}^\infty A_i=\bigcap_{i=1}^\infty \mathrm{Cl}_Y\, A_i\neq\emptyset$. Therefore, X is complete with respect to the development $\{\mathcal{H}_i\}_{i=1}^\infty$.

THEOREM 4. *A complete Moore space is a G_δ subset of every T_1-space in which it is dense and regularly embedded.*

PROOF. Let X be a Moore space which is complete with respect to the development $\{\mathcal{G}_i\}_{i=1}^\infty$. Let X be dense and regularly embedded in the T_1-space Y. For each natural number i and each point x in X, let $G_i(x)$ be an open subset of Y containing x such that $G_i(x)\cap X\in\mathcal{G}_i$ and $H_i(x)$ be an open subset of Y containing x such that $\mathrm{Cl}_Y\, H_i(x)\subset G_i(x)$. For each i, let $P_i=\bigcup\{H_i(x) : x\in X\}$. It is evident that $X\subset\bigcap_{i=1}^\infty P_i$. Let $p\in Y-X$ and suppose that $p\in\bigcap_{i=1}^\infty P_i$. For each i, there is a point $x_i\in X$ such that $p\in H_i(x_i)$. For each n, let $A_n=(\bigcap_{i=1}^n \mathrm{Cl}_Y\, H_i(x_i))\cap X$. Since X is dense in Y, each A_n is nonempty. Thus, $\{A_i\}_{i=1}^\infty$ is a nondecreasing sequence of nonempty closed sets in X such that, for each i, $A_i\subset G_n(x_n)$. Since X is complete with respect to the development $\{\mathcal{G}_i\}_{i=1}^\infty$, there is a point q common to all the A_i. Since X is regularly embedded in the T_1-space Y, there is an open subset U of Y containing q with $p\notin\mathrm{Cl}_Y\, U$. There is a natural number k such that, if $q\in G$ and $G\in\mathcal{G}_k$, then $G\subset U\cap X$. Since X is dense in Y, $H_k(x_k)\subset\mathrm{Cl}_Y(H_k(x_k)\cap X)$ and $H_k(x_k)\subset\mathrm{Cl}_Y\, U$. Thus, $p\notin H_k(x_k)$ and $X=\bigcap_{i=1}^\infty P_i$.

Theorem 4 can be strengthened by replacing the requirement that X be dense in Y with $\mathrm{Cl}_Y\, X$ is a G_δ subset of Y.

Any completely regular space is dense and, by the remark following Definition 2, regularly embedded in its Stone-Čech compactification. Hence, Theorems 3 and 4 have the following corollary.

COROLLARY 5. *A completely regular Moore space X is complete if and only if there is a bicompact Hausdorff space $Y \supset X$ such that X is a G_δ subspace of Y.*

If X is a T_1-space, let wX denote its Wallman compactification [1], [4], [7]. In [7], it is proved that wX is a bicompact T_1-space, but is not Hausdorff unless X is normal. We will show that, if X is regular, then X is regularly embedded in wX. Hence, any nonnormal T_3-space and its Wallman compactification is an example of a space which is regularly embedded in a nonregular space.

LEMMA 6. *If X is a regular T_1-space, p is a point of X and q is a point of wX distinct from p, then there are two disjoint open subsets of wX separating p and q.*

PROOF. Let $\mathfrak{Q}$ be the ultrafilter of closed sets in X which corresponds to q. Since p and q are distinct, there is a closed set $Q \in \mathfrak{Q}$ which does not contain p. Since X is regular, there are disjoint open subsets U and V of X containing p and Q, respectively. The open subsets $v(U)$ and $v(V)$ of wX induced by U and V, respectively, are disjoint and contain p and q, respectively.

THEOREM 7. *A regular space is regularly embedded in its Wallman compactification.*

Theorem 7 follows from the previous lemma by a modification of the proof that a bicompact Hausdorff space is regular.

Thus, by using the Wallman compactification instead of the Stone-Čech compactification we can remove the condition that the space be completely regular in Corollary 5.

THEOREM 8. *A Moore space is complete if and only if there is a bicompact T_1-space Y such that X is regularly embedded in Y and X is a G_δ subset of Y.*

Theorem 8 is a significant extension of Corollary 5, since there do exist Moore spaces which are not completely regular. In fact, there are examples of complete Moore spaces whose only real valued continuous functions are constant functions [6], [8].

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