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ON MAXIMAL GROUPS OF ISOMETRIES

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ABSTRACT. The purpose of this note is to introduce the concept of "Optimal Metrization" for metrizable topological spaces. Let X be such a space, ρ a metric on X and $K(\rho)$ the group of all those homeomorphisms of X onto itself which preserve ρ. The metric ρ is said to be "optimal" provided there is no ρ^* with $K(\rho^*)$ properly containing $K(\rho)$. A space having at least one optimal metric is called "optimally metrizable." Examples of spaces which are and which are not optimally metrizable are given; it is shown that the real line R is, and that the usual metric is optimal.

1. **Introduction and notation.** Let X be a metrizable topological space. We denote by $G(X)$ the group of all homeomorphisms of X onto itself and by $M(X)$ the set of all metrics on X compatible with the topology of X. We observe that with each $\rho \in M(X)$ there is associated the subgroup $K(\rho) \subseteq G(X)$ (group of all isometries for ρ) defined by $K(\rho) = \{h \mid h \in G(X)$ and $\rho(x, y) = \rho(h(x), h(y))$ for all $x, y \in X\}$. The basic idea motivating our investigations is the classification of a metric $\rho \in M(X)$ according to the size of the corresponding group $K(\rho)$.

CONVENTION. In this paper the set-theoretical inclusion is denoted by $\supseteq$, reserving $\supset$ for the *proper* inclusion.

DEFINITION 1.1. A metric $\rho \in M(X)$ is said to be *optimal* iff there is no $\rho^* \in M(X)$ with $K(\rho^*) \supset K(\rho)$. A space X is said to be *optimally metrizable* iff there is at least one optimal metric in $M(X)$.

Denoting by $L(X)$ the lattice of all subgroups of $G(X)$ we have the mapping $K: M(X) \to L(X)$ defined by $K(\rho) \in L(X)$ for $\rho \in M(X)$.

DEFINITION 1.2. The image of $M(X)$ under K is the subset $P(X) \subseteq L(X)$ partially ordered by inclusion. Its elements are groups of isometries and its maximal element (if it exists) is called a *maximal group of isometry*.

It is obvious that X is optimally metrizable if and only if $P(X)$ has a maximal element.

If $A \in L(X)$ and $h \in G(X)$ we denote by (A, h) the subgroup generated by A and h.

REMARK 1.1. It is obvious that a space X is not optimally metrizable iff to each $A \in P(X)$ there exists $h \in G(X)$ such that $h \notin A$ and $(A, h) \in P(X)$.

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2. **General properties of the set** $P(X)$. We observe that if $\rho \in M(X)$ then for each $g \in G(X)$ the function $g\rho$ defined by $g\rho(x, y) = \rho(g(x), g(y))$ for all $x, y \in X$ is again a metric $\in M(X)$; thus $G(X)$ acts on $M(X)$ in this natural way.

THEOREM 2.1. *For each* $\rho \in M(X)$ *and* $g \in G(X)$ *we have* $K(g\rho) = g^{-1}K(\rho)g$. *Thus* $P(X)$ *contains with each* $A \in P(X)$ *all its conjugates* $g^{-1}Ag \in P(X)$.

PROOF. By straightforward verification.

COROLLARY. *If* $\rho \in M(X)$ *is optimal then* $g\rho$ *is also optimal for every* $g \in G(X)$.

In the case X is compact we topologize $G(X)$ by the uniform convergence topology and it is a well-known fact (see for example [**1**] and [**2**]) that any compact subgroup K of $G(X)$ lies inside $K(\rho)$ for some ρ. Hence we obtain this obvious statement.

THEOREM 2.2. *If* X *is compact then* X *is optimally metrizable iff* $G(X)$ *has a maximal compact subgroup.*

3. **Optimal metrization property of some well-known spaces.** We first observe that any set X with the discrete topology is optimally metrizable and the optimal metric ρ is the most trivial one defined by $\rho(x, y) = 1$ for $x \neq y$. In this case we have $K(\rho) = G(X)$ and $P(X) = L(X)$. On the other hand we now show that the one-point compactification N^* of the set of positive integers N has *not* this property. Under N^* we understand the set $N \cup \{\infty\}$ metrized for example by: $\rho(n, m) = |1/n - 1/m|$ for $n, m \in N$ and $\rho(n, \infty) = 1/n$ for $n \in N$.

THEOREM 3.1. *The space* N^* *is not optimally metrizable.*

PROOF. Suppose that K were a maximal compact subgroup of $G(N^*)$. Any orbit $K(n) = \{g(n) \mid g \in K\}$ is compact; so if it were infinite, then it would include ∞. But no member of $G(N^*)$ moves ∞, so $K(n)$ is finite. Thus there are $n, m \in N$ with disjoint orbits $K(n)$, $K(m)$. Taking for h the simple transposition of m and n, we observe that the action of (K, h) differs from that of K only on the finite set $K(n) \cup K(m)$, and (K, h) is therefore again compact, which is impossible. Hence, N^* is not optimally metrizable.

THEOREM 3.2. *The compact interval* $[a, b]$ *is optimally metrizable and the usual metric* $|y - x|$ *is optimal.*

PROOF. We prove this showing that the group $K = \{e, r\}$ consisting of the identity e and the reflexion r ($r(x) = a + b - x$ for $x \in [a, b]$) is maximal compact in $G([a, b])$. If this were not the case then there would exist a larger compact group $K'\supset K$. If $h\in K'\backslash K$ we may assume h is increasing since $r\cdot h$ also belongs to $K'\backslash K$. Since $h\neq e$ there is $p\in[a, b]$ such that $h(p)\neq p$, and we know that there is an interval $[c, d]\subseteq[a, b]$ with p in the interior and the only points fixed by h are c and d. Now $\{h^n | n\geqq 1\}$ has a limit element g in K', so $g(c)$, $g(p)$, $g(d)$ are limit points of $\{h^n(c)\}$, $\{h^n(p)\}$, $\{h^n(d)\}$ respectively. Thus $g(c)=c$, $g(d)=d$ and $g(p)=c$ or d which is impossible. Hence $K'=K=\{e, r\}=K(\rho)$ where $\rho(x, y)=|y-x|$ showing that this metric is optimal.

THEOREM 3.3. *The circle S_1 is optimally metrizable.*

PROOF. Representing the circle S_1 in the form $S_1=\{e^{i\pi x}| x\in[-1, 1]\}$ we shall prove that the group $G\subseteq G(S_1)$ consisting of all rotations $e^{i\pi x}\rightarrow e^{i\pi(x+a)}$ and the reflexion $e^{i\pi x}\rightarrow e^{-i\pi x}$ is a maximal compact subgroup. To this end we observe that the set $G(S_1; -1)\subseteq G(S_1)$ of all those elements of $G(S_1)$ which leave invariant the point $-1=e^{-i\pi}$ is a *closed* subgroup of $G(S_1)$ which is homeomorphic and isomorphic to $G[-1, 1]$. If G^* were a compact group containing G, then according to Theorem 3.2 $G^*\cap G(S_1; -1)=G\cap G(S_1; -1)$. Now if g belongs to G^*, then we can find a rotation f such that $f(e^{-i\pi})=g(e^{-i\pi})$. Thus $h=f^{-1}g$ belongs to $G^*\cap G(S_1; -1)$, h belongs to G, $g=fh$ belong to G and $G^*=G$ which proves our assertion.

THEOREM 3.4. *The real line R is optimally metrizable and the usual metric $|y-x|$ is optimal.*

PROOF. The metric $|y-x|$ is preserved by the group of all translations and reflexions. Denoting this group by K, assume that there is $\rho\in M(R)$ with $K(\rho)\supset K$. Let $f\in K(\rho)\backslash K$. Without loss of generality we may assume f increasing and having at least one fixed point since otherwise we would apply on f suitable operations in K. Let F be the set of all fixed points of f. From $F\neq\emptyset$ we know that $R\backslash F$ has a connected component C that is an interval with at least one endpoint a. If we choose b in C, then $\{f^n(b)\}$ or $\{f^{-n}(b)\}$ will approach a, but $\rho(f^{n+1}(b), f^n(b))=\rho(f(b), b)=\rho(f^{-n-1}(b), f^{-n}(b))$ will not approach 0, which is impossible. Thus K is the maximal group of isometry corresponding to the metric $|y-x|$ which completes our proof.

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