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SEMIMODULAR POSETS AND THE JORDAN-DEDEKIND CHAIN CONDITION

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ABSTRACT. In this paper it is shown that an upper semimodular poset without infinite chains satisfies the Jordan-Dedekind chain condition. This corrects an error in Theorem 14, p. 40 of [1] and generalizes that theorem.

In [1, Theorem 14, p. 40], it is proved that the Jordan-Dedekind chain condition holds in any semimodular poset P of finite length. However, an error is made in the proof when the existence of a maximal chain γ'' connecting u to b is assumed. As can easily be shown by example, no such chain need exist. For instance, let $P=\{a, x_1, y_1 u, b, d\}$ be the six element poset in which x_1 and y_1 cover a, u and b cover x_1 and y_1, and d covers u and b. If, however, P is a lattice, then the above cited proof is correct as γ'' does exist. In this paper we give a correct proof to a generalization of the above theorem.

We say that a poset P is *upper semimodular* if whenever distinct elements $a, b \in P$ both cover an element $c \in P$ then there exists $d \in P$ which covers both a and b. In [1] it is assumed that P has a first element 0 and is of finite length. We do not need these restrictions here. A dual definition can be made for lower semimodular posets and it is easily seen that dual arguments prove that our theorems hold for lower semimodular as well as the stated upper semimodular posets. Definitions of all remaining terms may be found in [1]. If $a \in P$ we use the notation $U(a)=\{x \in P : x \geqq a\}$.

LEMMA 1. *Let P be an upper semimodular poset with no infinite chains. If $a \in P$, then $U(a)$ has a largest element $\hat{a}$.*

PROOF. Since P has no infinite chains, $U(a)$ has a maximal element $\hat{a}$. Suppose $b \neq \hat{a}$ is also a maximal element for $U(a)$. Let $A=[a, \hat{a}] \cap [a, b]$. Now $A \neq \varnothing$ since $a \in A$. Again since P has no infinite chains, A has a maximal element c_0. Now $c_0 < \hat{a}$, $c_0 < b$ since if $c_0 = \hat{a}$, say, then $\hat{a} \leqq b$ and hence $\hat{a} = b$ which is a contradiction. Let $\alpha: c_0 = a_0 < a_1 < \cdots < a_n = \hat{a}$ and $\beta: c_0 = b_0 < b_1 < \cdots < b_m = b$ be maximal chains from c_0 to $\hat{a}$ and b respectively. Proceeding by induction, for positive integers $1 \leqq i \leqq m$ let $P(i)$ be the proposition that there is a $c_i \in U(a)$ that covers b_i, $c_i \geqq a_1$ and $c_i \notin \beta$. Now a_1 and b_1 both cover c_0

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and $a_1 \neq b_1$ since if $a_1 = b_1$ then $a_1 \in A$ which contradicts the maximality of c_0. Thus by upper semimodularity there is a $c_1 \in P$ that covers a_1 and b_1. Now $c_1 \notin \beta$ since otherwise we would again have $a_1 \in A$. Therefore $P(1)$ holds. Now suppose $P(i-1)$ holds. By upper semimodularity there is a $c_i \in P$ that covers c_{i-1} and b_i. Hence, $c_i \geqq c_{i-1} \geqq a_1$ and $c_i \notin \beta$ since otherwise $a_1 \in A$. Thus, by induction, $P(i)$ holds for $1 \leqq i \leqq m$ and in particular $P(m)$ gives a $c_m \in U(a)$ that covers $b_m = b$ which contradicts the maximality of b. This shows that $\hat{a}$ is the unique maximal element for $U(a)$. It easily follows that $\hat{a}$ is the largest element of $U(a)$.

One sees from simple examples that the conclusion of Lemma 1 need not hold if either the chain restriction or the upper semimodularity are omitted. Also, as an immediate corollary, we have that *an upper semimodular poset with no infinite chains which has a least element also has a largest element.*

LEMMA 2. *Let P be an upper semimodular poset with no infinite chains. If $a \in P$ then every maximal chain from a to $\hat{a}$ has the same length.*

PROOF. Letting $b = \hat{a}$, the proof is the same as the proof of Theorem 14 of [1]. In this case the maximal chain γ'' from u to b exists since $u \leqq b$.

THEOREM 3. *An upper semimodular poset without infinite chains satisfies the Jordan-Dedekind chain condition.*

PROOF. Let $a < b$ and let α, β be maximal chains from a to b. Let γ be a maximal chain from b to $\hat{a}$. Applying Lemma 2 the length of the chain (α, γ) is the same as the length of the chain (β, γ). Hence the length of α equals the length of β.

COROLLARY 4. *In a poset P in which every interval lies in an upper semimodular subposet without infinite chains (maximal chains being the same in P and the subposets), the Jordan-Dedekind chain condition is satisfied.*

Although an interval of an upper semimodular lattice is upper semimodular, one can give simple examples to show this need not hold in upper semimodular posets. This observation and the previous corollary are due to the referee.

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