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INTEGRAL RING EXTENSIONS AND PRIME IDEALS OF INFINITE RANK

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ABSTRACT. An example is constructed showing that for an integral ring extension $R\subset T$, and a prime ideal P of R having infinite rank, it can happen that in T each prime ideal lying over P has finite rank.

By the rank (or height) of a prime ideal P in a commutative ring R is meant the maximal length of descending chains of prime ideals of R starting with P. Thus P has rank n if there exists a descending chain $P=P_0\supset P_1\supset \cdots \supset P_n$, but no such chain of longer length; and P has infinite rank (or rank ∞) if there exist arbitrarily long chains of primes descending from P. Let $R\subset T$ be a pair of commutative rings (having a common identity). One says that the going up property (GU) holds for the pair $R\subset T$ if whenever $P\subset P_0$ are prime ideals in R and Q is a prime of T such that $Q\cap R=P$, then there exists a prime Q_0 in T such that $Q\subset Q_0$ and $Q_0\cap R=P_0$. It is well known that if T is integral over R, then GU holds for the pair $R\subset T$; and it can be readily seen that if $R\subset T$ satisfies GU and P is a prime ideal in R of rank n, then there exists in T a prime ideal Q such that Q has rank $\geqq n$ and $Q\cap R=P$ [**3**, Theorem 46, p. 31]. We show, however, that this result cannot be extended to primes of rank ∞ even for R an integral domain and T the integral closure of R. Of course GU insures that there can be no fixed bound on the ranks of the primes of T lying over a rank ∞ prime P of R. Thus in our example there must be infinitely many primes of T lying over P. In particular, T cannot be a finite R-module [**1**, p. 40].

The idea involved in our construction may be stated as follows.

LEMMA. *Let R be a quasi-local domain with maximal ideal P and quotient field K. Assume that for each positive integer n there exists a valuation ring of K containing R and having rank n, but that R is contained in no valuation ring of K having infinite rank. Let T be the integral closure of R. If T is a Prüfer domain, then P has infinite rank but each prime ideal of T has finite rank.*

PROOF. If Q is a prime ideal of T, then the localization T_Q is a

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valuation ring and the rank of the prime ideal Q equals the rank of the valuation ring T_Q. Thus each prime of T has finite rank and by intersecting chains of primes of T with R, we see that P has infinite rank.

CONSTRUCTION OF THE EXAMPLE. Let k be an arbitrary field and let $\{x_i\}_{i=1}^{\infty}$ be a collection of indeterminates over k. We construct a rank one valuation ring V_1 on the field $K=k(x_1, x_2, \cdots)$ such that V_1 has the form $k+M_1$ where M_1 is the maximal ideal of V_1. This can be done, for example, by mapping the x_i onto rationally independent real numbers and then extending this map to a valuation of K trivial on k. The x_i having rationally independent values assures that k maps isomorphically onto the residue field of V_1 and hence that $V_1=k+M_1$. For each integer $n\geqq 2$, let L_n denote the field $k(\{x_i \mid i\leqq n \text{ or } i\geqq 2n\})$. Thus $K=L_n(x_{n+1}, \cdots, x_{2n-1})$ and $x_{n+1}, \cdots, x_{2n-1}$ are algebraically independent over L_n. Consider the valuation ring $V_1\cap L_n$. By mapping $x_{n+1}, \cdots, x_{2n-1}$ onto suitably chosen elements of a suitable totally ordered abelian group containing the value group of V_n we can obtain a valuation ring V_n of K such that:

(1) $V_n\cap L_n=V_1\cap L_n$.

(2) V_n has rank n.

(3) V_n has the form $k+M_n$ where M_n is the maximal ideal of V_n.

See, for example, [1, Proposition 1, p. 161].

Let $P=\bigcap_{i=1}^{\infty} M_i$ and let $R=k+P$. We note that R is a quasi-local domain with maximal ideal P. For if α is a nonzero element of k and $m\in P$, then $(\alpha+m)^{-1}=\alpha^{-1}+m'$, where $m'=-m/\alpha(\alpha+m)\in M_i$ for each i, so $m'\in P$. Let T be the integral closure of R.

CLAIM. T is a Prüfer domain with quotient field K, $T=\bigcap_{i=1}^{\infty} V_i$, and no valuation ring between T and K has infinite rank.

PROOF. Let $K_n=k(x_1, \cdots, x_n)$, $R_n=R\cap K_n$, and let T_n be the integral closure of R_n. Note that for $s\geqq n$, $V_s\cap K_n=V_1\cap K_n$. Hence

$$R_n = k + \left(\bigcap_{i=1}^{n-1} M_i \cap K_n\right).$$

We show that $T_n=\bigcap_{i=1}^{n-1} V_i\cap K_n$. If $y\in\bigcap_{i=1}^{n-1} V_i\cap K_n$ then there exists $a_i\in k$ such that $y-a_i\in M_i$, for each i such that $1\leqq i<n$. It follows that $\prod_{i=1}^{n-1}(y-a_i)\in\bigcap_{i=1}^{n-1} M_i\cap K_n\subset R_n$ so y satisfies an equation of integral dependence over R_n. Thus T_n is a finite intersection of valuation rings of the field K_n. Hence T_n is a Prüfer domain with quotient field K_n and each valuation ring containing T_n contains some $V_i\cap K_n$ [1, p. 132–134]. It follows that $T=\bigcup_{i=1}^{\infty} T_i$ is also Prüfer [2, p. 260], T has quotient field K, and $T=\bigcap_{i=1}^{\infty} V_i$. Now

suppose W is a valuation ring between T and K. Since W contains T_n, W contains some $V_i \cap K_n$. If W contains $V_1 \cap K_n$ for all n, then W contains V_1 so either $W = V_1$ or $W = K$. If $V_1 \cap K_n \not\subset W$, then for $s \geqq n$, let $W_s = W \cap K_s$. We know that $V_j \cap K_s \subset W_s$ for some $j < s$. But, for $j \geqq n$, $V_j \cap K_s \cap K_n = V_1 \cap K_n$, so W_s is contained in $V_j \cap K_s$ for some $j < n$. Since V_j has rank j, we see that W_s has rank $< n$. It follows that $W = \bigcup_{s=n}^{\infty} W_s$ also has rank less than n.

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