IDEALS IN THE MODULAR GROUP RING
OF A $p$-GROUP

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Abstract. We show that if $G$ has order $p^n$ then the group ring has a chain of $p^n+1$ ideals and that the radical powers are canonical in the lattice of ideals. We then prove that if $G$ is abelian, $G$ is determined by the lattice of ideals.

This paper concerns the lattice of ideals in the group ring of a finite $p$-group over the integers modulo $p$, for $p$ a prime. This field is written as $K$ and the group ring as $KG$. In [1] it is shown that if $G$ and $H$ are abelian $p$-groups such that $KG$ is isomorphic to $KH$, then $G$ is isomorphic to $H$. We extend this result to the following:

Theorem. If $G$ and $H$ are abelian $p$-groups such that the lattice of ideals of $KG$ is isomorphic to the lattice of ideals of $KH$, then $G$ is isomorphic to $H$.

Let $\mathfrak{N}$ be the radical of $KG$ and $\mathfrak{V}$ be a vector space in $KG$ such that $\mathfrak{N}^{w+1} \subseteq \mathfrak{V} \subseteq \mathfrak{N}^w$. If $\alpha$ is in $\mathfrak{V}$ and $g$ is a member of $G$, then $g\alpha = \alpha g \equiv \alpha \mod \mathfrak{N}^w$ so that $\mathfrak{V}$ is an ideal in $KG$. Hence if $\mathfrak{N}^w/\mathfrak{N}^{w+1}$ has dimension $t_w$, the lattice of ideals which are contained in $\mathfrak{V}$ and contain $\mathfrak{N}^{w+1}$ is isomorphic to the lattice of subvector spaces of the vector space of dimension $t_w$ over $K$. Therefore, if $G$ has order $p^n$, $KG$ has a chain of $p^n+1$ ideals. By the modularity of the lattice of ideals, each ideal of dimension $m$, for $0 < m < p^n$, contains an ideal of dimension $m-1$ and is contained in an ideal of dimension $m+1$.

Lemma 1. If $\mathfrak{S}$ and $\mathfrak{S}'$ are ideals in $KG$ such that $\mathfrak{S}$ covers $\mathfrak{S}'$, then $\alpha(g-1)$ is in $\mathfrak{S}'$ for all $\alpha$ in $\mathfrak{S}$ and $g$ in $G$.

Proof. If $\mathfrak{S}$ covers $\mathfrak{S}'$, then $\mathfrak{S}/\mathfrak{S}'$ has dimension one. If $\alpha$ is in $\mathfrak{S}$ and not in $\mathfrak{S}'$, then the members of $\mathfrak{S}/\mathfrak{S}'$ are $k\alpha + \mathfrak{S}'$ for $k$ in $K$. Hence for each $g$ in $G$, $ag \equiv k\alpha \mod \mathfrak{S}'$ for some $k$. If $k \neq 1$, then $g-k$ is a unit in $KG$ so that $\alpha(g-k) \equiv 0 \mod \mathfrak{S}'$ implies $\alpha \equiv 0 \mod \mathfrak{S}'$. Therefore, $\alpha(g-1)$ is in $\mathfrak{S}'$ for all $\alpha$ in $\mathfrak{S}$ and $g$ in $G$.

Lemma 2. The intersection of the ideals covered by $\mathfrak{N}$ is $\mathfrak{N}^{w+1}$.

Proof. Let $\mathfrak{N}^w/\mathfrak{N}^{w+1}$ have dimension $t_w$ and let $N_1^w, \ldots, N_{t_w}^w$ be a basis for $\mathfrak{N}^w/\mathfrak{N}^{w+1}$. For each fixed $j$ such that $1 \leq j \leq t_w$, let $\mathfrak{S}_j$ be the
collection of members of $KG$ of the form $\sum a_i N_i^w + \mathfrak{R}_w^{u+1}$ with $a_i = 0$. Clearly $\mathfrak{s}_j$ is an ideal, $\mathfrak{R}_w$ covers $\mathfrak{s}_j$, and the intersection of the $\mathfrak{s}_j$ as $j$ ranges from 1 to $t_w$ is $\mathfrak{R}_w^{u+1}$. Hence the intersection is contained in $\mathfrak{R}_w^{u+1}$.

By Lemma 1, if $\mathfrak{s}$ is any ideal which is covered by $\mathfrak{R}_w$, then $\alpha(g - 1)$ is in $\mathfrak{s}$ for all $g$ in $G$ and $\alpha$ in $\mathfrak{R}_w$. Since $\mathfrak{R}_w^{u+1}$ is generated by elements of the form $\alpha(g - 1)$, $\mathfrak{R}_w^{u+1}$ is contained in any ideal which is covered by $\mathfrak{R}_w$. Hence $\mathfrak{R}_w^{u+1}$ is contained in the intersection and the lemma is proved.

Using the results of [2] it can be shown that the dual of Lemma 2 holds; that is, the join of the ideals which cover $\mathfrak{R}_w^{u+1}$ is $\mathfrak{R}_w^w$.

The $\mathfrak{M}$-series for $G$ [3] is defined as follows: $\mathfrak{M}_1 = G$; for $i > 1$, $\mathfrak{M}_i = \langle [\mathfrak{M}_{i-1}, G], \mathfrak{M}_{i-1}^p \rangle$ where $(i/p)$ is the least integer not greater than $i/p$ and $\mathfrak{M}_i^p$ is the set of all $p$th powers of members of $\mathfrak{M}_i$.

**Lemma 3.** If the lattice of ideals of $KG$ is isomorphic to the lattice of ideals of $KH$, then for each $i$, $\mathfrak{M}_i(G)/\mathfrak{M}_i^{u+1}(G)$ is isomorphic to $\mathfrak{M}_i(H)/\mathfrak{M}_i^{u+1}(H)$.

**Proof.** By Lemma 2, $\mathfrak{R}_w$ is canonical in the lattice of ideals; therefore $t_w$, the dimension of $\mathfrak{R}_w/\mathfrak{R}_w^{u+1}$, is determined by the lattice of ideals. By [3, Theorem 3.7], determining all the $t_i$ is equivalent to determining the $d_i$, where $\mathfrak{M}_i/\mathfrak{M}_i^{u+1}$ has order $p^{d_i}$. Since $\mathfrak{M}_i/\mathfrak{M}_i^{u+1}$ is elementary abelian, the quotient is determined by $d_i$.

The proof of the theorem is immediate since, as noted in [4], an abelian group is determined by its $\mathfrak{M}$-series.

**References**


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