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IDEALS IN THE MODULAR GROUP RING OF A p-GROUP

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ABSTRACT. We show that if G has order p^n then the group ring has a chain of p^n+1 ideals and that the radical powers are canonical in the lattice of ideals. We then prove that if G is abelian, G is determined by the lattice of ideals.

This paper concerns the lattice of ideals in the group ring of a finite p-group over the integers modulo p, for p a prime. This field is written as K and the group ring as KG. In [1] it is shown that if G and H are abelian p-groups such that KG is isomorphic to KH, then G is isomorphic to H. We extend this result to the following:

THEOREM. *If G and H are abelian p-groups such that the lattice of ideals of KG is isomorphic to the lattice of ideals of KH, then G is isomorphic to H.*

Let $\mathfrak{N}$ be the radical of KG and $\mathcal{V}$ be a vector space in KG such that $\mathfrak{N}^{w+1}\subseteq\mathcal{V}\subseteq\mathfrak{N}^w$. If α is in $\mathcal{V}$ and g is a member of G, then $g\alpha\equiv\alpha g\equiv\alpha \bmod \mathfrak{N}^w$ so that $\mathcal{V}$ is an ideal in KG. Hence if $\mathfrak{N}^w/\mathfrak{N}^{w+1}$ has dimension t_w, the lattice of ideals which are contained in $\mathfrak{N}^w$ and contain $\mathfrak{N}^{w+1}$ is isomorphic to the lattice of subvector spaces of the vector space of dimension t_w over K. Therefore, if G has order p^n, KG has a chain of p^n+1 ideals. By the modularity of the lattice of ideals, each ideal of dimension m, for $0<m<p^n$, contains an ideal of dimension $m-1$ and is contained in an ideal of dimension $m+1$.

LEMMA 1. *If $\mathcal{I}$ and $\mathcal{I}'$ are ideals in KG such that $\mathcal{I}$ covers $\mathcal{I}'$, then $\alpha(g-1)$ is in $\mathcal{I}'$ for all α in $\mathcal{I}$ and g in G.*

PROOF. If $\mathcal{I}$ covers $\mathcal{I}'$, then $\mathcal{I}/\mathcal{I}'$ has dimension one. If α is in $\mathcal{I}$ and not in $\mathcal{I}'$, then the members of $\mathcal{I}/\mathcal{I}'$ are $k\alpha+\mathcal{I}'$ for k in K. Hence for each g in G, $\alpha g\equiv k\alpha \bmod \mathcal{I}'$ for some k. If $k\neq 1$, then $g-k$ is a unit in KG so that $\alpha(g-k)\equiv 0 \bmod \mathcal{I}'$ implies $\alpha\equiv 0 \bmod \mathcal{I}'$. Therefore, $\alpha(g-1)$ is in $\mathcal{I}'$ for all α in $\mathcal{I}$ and g in G.

LEMMA 2. *The intersection of the ideals covered by $\mathfrak{N}^w$ is $\mathfrak{N}^{w+1}$.*

PROOF. Let $\mathfrak{N}^w/\mathfrak{N}^{w+1}$ have dimension t_w and let $N_1^w, \cdots, N_{t_w}^w$ be a basis for $\mathfrak{N}^w/\mathfrak{N}^{w+1}$. For each fixed j such that $1\leqq j\leqq t_w$, let $\mathcal{I}_j$ be the

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collection of members of KG of the form $\sum a_i N_i^w + \mathfrak{N}^{w+1}$ with $a_j = 0$. Clearly $\mathcal{I}_j$ is an ideal, $\mathfrak{N}^w$ covers $\mathcal{I}_j$, and the intersection of the $\mathcal{I}_j$ as j ranges from 1 to t_w is $\mathfrak{N}^{w+1}$. Hence the intersection is contained in $\mathfrak{N}^{w+1}$.

By Lemma 1, if $\mathcal{I}$ is any ideal which is covered by $\mathfrak{N}^w$, then $\alpha(g-1)$ is in $\mathcal{I}$ for all g in G and α in $\mathfrak{N}^w$. Since $\mathfrak{N}^{w+1}$ is generated by elements of the form $\alpha(g-1)$, $\mathfrak{N}^{w+1}$ is contained in any ideal which is covered by $\mathfrak{N}^w$. Hence $\mathfrak{N}^{w+1}$ is contained in the intersection and the lemma is proved.

Using the results of [2] it can be shown that the dual of Lemma 2 holds; that is, the join of the ideals which cover $\mathfrak{N}^{w+1}$ is $\mathfrak{N}^w$.

The $\mathfrak{M}$-series for G [3] is defined as follows: $\mathfrak{M}_1 = G$; for $i > 1$, $\mathfrak{M}_i = \langle [\mathfrak{M}_{i-1}, G], \mathfrak{M}^p_{(i/p)} \rangle$ where (i/p) is the least integer not greater than i/p and $\mathfrak{M}_k^p$ is the set of all pth powers of members of $\mathfrak{M}_k$.

LEMMA 3. *If the lattice of ideals of KG is isomorphic to the lattice of ideals of KH, then for each i, $\mathfrak{M}_i(G)/\mathfrak{M}_{i+1}(G)$ is isomorphic to $\mathfrak{M}_i(H)/\mathfrak{M}_{i+1}(H)$.*

PROOF. By Lemma 2, $\mathfrak{N}^w$ is canonical in the lattice of ideals; therefore t_w, the dimension of $\mathfrak{N}^w/\mathfrak{N}^{w+1}$, is determined by the lattice of ideals. By [3, Theorem 3.7], determining all the t_w is equivalent to determining the d_i, where $\mathfrak{M}_i/\mathfrak{M}_{i+1}$ has order p^{d_i}. Since $\mathfrak{M}_i/\mathfrak{M}_{i+1}$ is elementary abelian, the quotient is determined by d_i.

The proof of the theorem is immediate since, as noted in [4], an abelian group is determined by its $\mathfrak{M}$-series.

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