

THE SELFADJOINT OPERATORS OF A VON NEUMANN  
 ALGEBRA FORM A CONDITIONALLY  
 COMPLETE LATTICE

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ABSTRACT. The bounded resolutions of the identity in a von Neumann algebra can be ordered by  $\{E_S(u)\} \preceq \{E_T(u)\}$  if  $E_S(u) \geq E_T(u)$ ,  $u \in R$ . The selfadjoint operators in the algebra are partially ordered by this relation and are shown to form a conditionally complete lattice. The lattice operations are (essentially) defined by  $E_{\vee S_\alpha}(u) = \bigwedge E_{S_\alpha}(u)$  for all  $u$  contained in  $R$ . This order is called spectral order and agrees with the usual order on commutative subalgebras. For positive operators,  $S$  is greater than or equal to  $T$  in spectral order if and only if  $S^n$  is greater than or equal to  $T^n$  in the usual order for all  $n \geq 1$ . Kadison's well-known counterexample is shown to fail. The operator lattice defined by spectral order differs from a vector lattice in the fact that  $S \succeq T$  does not imply that  $S+C \succeq T+C$ .

1. **Introduction.** Let  $\mathfrak{A}$  be a von Neumann algebra of operators on a Hilbert space  $H$ , and let  $\mathfrak{S}$  be the real vector space of selfadjoint elements of  $\mathfrak{A}$ .  $S \leq T$  means

$$(1.1) \quad (Sx, x) \leq (Tx, x)$$

for all  $x \in H$ . With this partial order,  $\mathfrak{S}$  is a partially ordered vector space. Kadison [1] showed, long ago, that even when  $\mathfrak{A}$  is the set of all bounded operators on  $H$ ,  $\mathfrak{S}$  is not a complete vector lattice. In fact, Sherman [3] showed that if the selfadjoint operators in a  $C^*$ -algebra are lattice-ordered by the nonnegative elements, then the algebra is commutative.

The purpose of this paper is to report on an investigation of another order relation on  $\mathfrak{S}$  with respect to which  $\mathfrak{S}$  is conditionally complete. This order, called spectral order and denoted by " $\succeq$ ", is not a vector order on  $\mathfrak{S}$ . The missing spectral property is that

$$(1.2) \quad A - B \succeq 0$$

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does not imply

$$(1.3) \quad A \succeq B$$

unless  $A$  and  $B$  commute. In that case,  $A \succeq B$  is equivalent to  $A \succcurlyeq B$ . For positive operators,  $A \succcurlyeq B$  if and only if  $A^p \succeq B^p$  for all integers  $p \geq 1$ . This paper is a revision and extension of another paper having the same title which was circulated privately.

**2. Résolutions of the identity.** Let  $\mathcal{O}$  be the selfadjoint projections in  $\mathcal{A}$ . It is known that each element of  $\mathcal{S}$  finds its spectral resolution in  $\mathcal{O}$  and that  $\mathcal{O}$  is a complete sublattice of the complete lattice of all projections in  $H$ . The latter statement means that the supremum of a family in  $\mathcal{O}$  in the lattice of all projections is an element of  $\mathcal{O}$ . A resolution of the identity in  $\mathcal{A}$  is a right-continuous isotone map  $T$  of the line  $R$  into  $\mathcal{O}$  such that there exists  $m > 0$  such that

$$(2.1) \quad T(u) = 0, \quad u < -m \quad \text{and}$$

$$(2.2) \quad T(u) = I, \quad u > m.$$

By the uniqueness assertion of the spectral theorem, it is known that  $\mathcal{C}$ , the set of resolutions of the identity in  $\mathcal{A}$ , is in one-to-one correspondence with  $\mathcal{S}$ .

**DEFINITION 1.** Let  $S_1$  and  $S_2$  be selfadjoint elements of the von Neumann algebra  $\mathcal{A}$ . Let  $T_1$  and  $T_2$  be the resolutions of the identity of  $S_1$  and  $S_2$ , respectively.  $S_1 \preceq S_2$  if

$$(2.3) \quad T_1(u) \geq T_2(u), \quad u \in R.$$

The order " $\preceq$ " will be called spectral order. (It is clear that spectral order is partial order in  $\mathcal{S}$ .)

**LEMMA 1.** Let  $\{T_\alpha\}_{\alpha \in A} \subset \mathcal{C}$  and suppose there exists  $T \in \mathcal{C}$  such that

$$(2.4) \quad T(u) \leq T_\alpha(u), \quad u \in R, \alpha \in A.$$

Define

$$(2.5) \quad W(u) = \bigwedge_{\alpha \in A} T_\alpha(u), \quad u \in R.$$

Then  $W \in \mathcal{C}$ . If  $Y \in \mathcal{C}$  is such that

$$(2.6) \quad W(u) \leq Y(u) \leq T_\alpha(u), \quad u \in R, \alpha \in A,$$

then

$$(2.7) \quad W = Y.$$

**PROOF.** Let  $m$  be the bound on  $T$  required in (2.1) and (2.2). Let  $m_\alpha$  be the bound for  $T_\alpha$  for some  $\alpha \in A$ . Then,  $W(u) = 0$ ,  $u < -m_\alpha$ , and

$W(u) = I$  for  $u > m$ . Thus,  $W$  has a bound  $\max\{m, m_\alpha\}$ .  $W$  is clearly an isotone map of  $R$  into  $\mathcal{O}$ .

$W$  is right-continuous. For,

$$(2.8) \quad \begin{aligned} \bigwedge_{v > u} W(v) &= \bigwedge_{v > u} \left( \bigwedge_{\alpha \in A} T_\alpha(v) \right) = \bigwedge_{\alpha \in A} \left( \bigwedge_{v > u} T_\alpha(v) \right) \\ &= \bigwedge_{\alpha \in A} T_\alpha(u) = W(u). \end{aligned}$$

Thus,  $W \in \mathcal{C}$ . The last assertion is clear from the definition of  $W$ . Q.E.D.

If  $T$  is any isotone map of  $R$  into  $\mathcal{O}$ , then  $T$  is called bounded if it satisfies (2.1) and (2.2) for some finite number  $m > 0$ .

LEMMA 2. *If  $T$  is a bounded isotone map of  $R$  into  $\mathcal{O}$ , then  $T$  determines a unique element  $T_r \in \mathcal{C}$  by defining*

$$(2.9) \quad T_r(u) = \bigwedge_{v > u} T(v).$$

$T_r$  is called the right-regularization of  $T$ .

PROOF. Let  $\epsilon > 0$  be any positive number. If  $m$  is a bound for  $T$ , then  $m + \epsilon$  is a bound for  $T_r$ .  $T_r$  is an isotone map of  $R$  into  $\mathcal{O}$ . It remains to show that  $T_r$  is right-continuous.

$$(2.10) \quad \bigwedge_{v > u} T_r(v) = \bigwedge_{v > u} \bigwedge_{w > v} T(w) = \bigwedge_{w > u} T(w) = T_r(u).$$

$T_r \in \mathcal{C}$ . Q.E.D.

LEMMA 3. *Let  $\{T_\alpha\}_{\alpha \in A} \subset \mathcal{C}$  and suppose there exists  $T \in \mathcal{C}$  such that*

$$(2.11) \quad T(u) \geq T_\alpha(u), \quad u \in R, \alpha \in A.$$

Define

$$(2.12) \quad V(u) = \bigvee_{\alpha \in A} T_\alpha(u), \quad u \in R.$$

Then  $V$  is a bounded isotone map of  $R$  into  $\mathcal{O}$ .  $V_r$ , the right-regularization of  $V$ , is contained in  $\mathcal{C}$  and has the property that if  $Y \in \mathcal{C}$  and

$$(2.13) \quad V_r(u) \geq Y(u) \geq T_\alpha(u), \quad u \in R, \alpha \in A,$$

then

$$(2.14) \quad V_r = Y.$$

PROOF.  $V$  is a bounded isotone map by arguments analogous to those used in the proof of Lemma 1.  $V_r \in \mathcal{C}$  by Lemma 2.  $V_r(u) \geq V(u)$

by the definition of the right-regularization given in equation (2.9). It follows that

$$(2.15) \quad V_r(u) \geq Y(u) \geq V(u), \quad u \in R.$$

But then, for all  $u \in R$ ,

$$(2.16) \quad Y(u) \leq V_r(u) = \bigwedge_{v>u} V(v) \leq \bigwedge_{v>u} Y(v) = Y(u).$$

So  $Y = V_r$ . Q.E.D.

**THEOREM 1.**  $\mathcal{S}$  is a conditionally complete lattice under the partial order " $\succeq$ ".

**PROOF.** In Lemmas 1, 2, and 3, it has been shown that  $\mathcal{C}$  is a conditionally complete lattice under the reverse of the product order determined by the order structure in  $\mathcal{O}$ .  $\mathcal{C}$  is in one-to-one correspondence with  $\mathcal{S}$ ; and the order " $\succeq$ " in  $\mathcal{S}$  is by definition the aforementioned order in  $\mathcal{C}$ . Q.E.D.

**3. Properties of spectral order.** The purpose of this section is to compare and contrast spectral order with the usual order on self-adjoint operators. An element  $T \in \mathcal{S}$  is nonnegative in spectral order if  $T \succeq 0$ . By definition, the spectrum of  $T$  is nonnegative; and, hence,  $T \geq 0$  in the usual order. Conversely,  $T \geq 0$  implies  $T$  has nonnegative spectrum and so  $T \succeq 0$ . The two orders, therefore, have the same positive cones.

It is known that  $(\mathcal{S}, \geq)$  is an ordered vector space, and that the order is completely determined by the positive cone. If  $(\mathcal{S}, \succeq)$  were an ordered vector space, then, since it shares the same positive cone with  $(\mathcal{S}, \geq)$ , it would be the same order. We shall show below that  $(\mathcal{S}, \succeq)$  is not an ordered vector space and that " $\succeq$ " is distinct from " $\geq$ ".

**LEMMA 4.** Let  $S, T \in \mathcal{S}$  and  $S \succeq T$ . Then  $S \geq T$ .

**PROOF.** Let

$$(3.1) \quad f_n = \sum_{i=1}^{p_n} \alpha_{in} \chi_{E_{in}}$$

be a sequence of nonnegative nondecreasing simple functions converging monotonely upward to the function  $f(u) = u$ . We may assume that these simple functions are constructed on a partition of  $[-m, m]$  into subintervals of length  $2m/n$ , where  $m$  is some number greater

than both  $\|S\|$  and  $\|T\|$ . Then, clearly,

$$(3.2) \quad f_n(S) \uparrow S$$

and

$$(3.3) \quad f_n(T) \uparrow T.$$

But

$$(3.4) \quad (f_n(T)x, x) \leq (f_n(S)x, x), \quad x \in H,$$

as direct calculation easily shows. Hence,

$$(3.5) \quad (Sx, x) \geq (f_n(S)x, x) \geq (f_n(T)x, x), \quad n = 1, 2, \dots$$

Taking the limit on the right-hand side, we have

$$(3.6) \quad (Sx, x) \geq (Tx, x), \quad x \in H. \quad \text{Q.E.D.}$$

**THEOREM 2.** *Let  $S, T \in \mathfrak{s}$ . If  $S$  and  $T$  commute,  $S \succeq T$  is equivalent to  $S \geq T$ .*

**PROOF.** By Lemma 4,  $S \succeq T$  implies  $S \geq T$ . Conversely, suppose  $S \geq T$ . Since  $S$  and  $T$  commute by hypothesis, the families  $\{E_S(u)\}_{u \in R}$  and  $\{E_T(u)\}_{u \in R}$  commute. If  $S \succeq T$  does not hold, there exists  $u \in R$  such that

$$(3.7) \quad E_S(u)E_T(u) = E_T(u)E_S(u) < E_S(u).$$

This means that there exists an  $x \neq 0$  such that

$$(3.8) \quad x \in E_S(u)(I - E_T(u))H.$$

According to Theorem 5 in Lengyel and Stone's paper [2],

$$(3.9) \quad (Sx, x) \leq u(x, x)$$

since  $u$  is the least upper bound of the form  $(Sx, x)$  on  $E_S(u)H$ . On the other hand,

$$(3.10) \quad (Tx, x) > u(x, x)$$

because  $x \in (I - E_T(u))H$  and  $u$  is the greatest lower bound of the form  $(Tx, x)$ . Q.E.D.

Finally, spectral order can be characterized in terms of the usual order.

**THEOREM 3.** *Let  $A$  and  $B$  be bounded positive selfadjoint operators on  $H$ . Then  $A \succeq B$  if and only if  $A^p \geq B^p$  for  $p = 1, 2, \dots$*

**PROOF.** According to Lengyel and Stone [2], if  $A$  is a bounded

positive selfadjoint operator, then for each  $u \in R$  the spectral projection  $E_A(u)$  is the projection on the closed linear manifold

$$(3.12) \quad \mathfrak{M}_A(u) = \{x \mid \|A^p x\| \leq u^p \|x\|, p \geq 1\}.$$

But,

$$(3.13) \quad \|A^p x\|^2 = (A^{2p} x, x) \geq (B^{2p} x, x) = \|B^p x\|^2.$$

Hence,

$$(3.14) \quad \mathfrak{M}_B(u) \supset \mathfrak{M}_A(u),$$

and

$$(3.15) \quad E_B(u) \geq E_A(u), \quad u \in R.$$

This implies, by definition,

$$(3.16) \quad A \succeq B.$$

Conversely, suppose  $A \succeq B$ . We wish to show that

$$(3.17) \quad \int u^p (E_A(du)x, x) \geq \int u^p (E_B(du)x, x)$$

for all  $x \in H$  and  $p = 1, 2, \dots$ . Exactly the same sequence of simple functions mentioned in the proof of Lemma 4 can be employed to show that

$$(3.18) \quad (f_n^p(A)x, x) \uparrow (A^p x, x) \text{ and}$$

$$(3.19) \quad (f_n^p(B)x, x) \uparrow (B^p x, x)$$

for all  $x \in H$  and  $p = 1, 2, \dots$ . Because of the nondecreasing character of  $f_n^p$  as a function of  $u \in R$ , it is easy to show by direct calculation that

$$(3.20) \quad (f_n^p(A)x, x) \geq (f_n^p(B)x, x)$$

for  $x \in H$  and  $p = 1, 2, \dots$ . (3.17) follows by taking limits on  $n$ . Q.E.D.

**COROLLARY 1.** *Let  $A$  and  $B$  be bounded positive selfadjoint operators on  $H$ . Let  $D$  be a bounded closed interval which contains the spectra of both  $A$  and  $B$ . Then  $A \succeq B$  if and only if for all  $f$  continuous and nondecreasing on  $D$ ,  $f(A) \geq f(B)$ .*

**PROOF.**  $u^p$  is continuous and nondecreasing on  $D$  for  $p = 1, 2, \dots$ . This implies the sufficiency of the condition  $f(A) \geq f(B)$ . The proof of necessity follows by exactly the same approximation argument used in the proof of the theorem. Q.E.D.

Finally, let us show that spectral order and the usual order are

distinct. The following example is essentially due to Kadison [1]. Let  $H$  be a three-dimensional complex Hilbert space with orthogonal basis  $\{e_1, e_2, e_3\}$ . Let  $M$  be the closed subspace spanned by  $e_1$  and  $e_2$ .  $P_i$  is projection on  $e_i$  for  $i=1, 2, 3$ .  $Q$  is projection on  $M$ . Let  $A$  be the operator on  $M$  whose matrix is given by

$$(3.21) \quad A = \begin{pmatrix} -1 & 2^{1/2} \\ 2^{1/2} & -1 \end{pmatrix}.$$

The selfadjoint operator  $AQ$  is neither positive nor negative in the usual order of selfadjoint operators. Thus, it is neither bounded above nor below by the zero operator.  $P_1$  and  $P_2$  are commuting orthogonal projections. By the remark above, they have infimum 0 in spectral order. Suppose that 0 was the infimum in the usual order. A contradiction is reached if it can be shown that  $AQ$  is less than  $P_1$  and  $P_2$  in the usual order; for, in that case, 0 is not the greatest of the lower bounds since  $AQ$  and 0 are not even comparable.

In matrix form on  $H$ ,

$$(3.22) \quad P_1 - AQ = \begin{pmatrix} 2 & 2^{1/2} & 0 \\ 2^{1/2} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By the usual test, this is a positive matrix and so defines a positive operator. In fact, the eigenvalues are  $\lambda=0$  and  $\lambda=3$ . Because the usual order is a vector order  $P_1 - AQ \geq 0$  if and only if  $P_1 \geq AQ$ . Similar calculations show that  $P_2 \geq AQ$ . Therefore, 0 is not the infimum in the usual order and so the two orders are distinct.

By remarks above, it is certainly true that  $P_1 - AQ \succeq 0$ . Let us show that  $P_1 \succeq AQ$  does *not* hold. The spectral resolution of  $P_1$  is given by

$$(3.23) \quad E(u) = 0, \quad -\infty < u < 0;$$

$$(3.24) \quad E(u) = P_2 + P_3, \quad 0 \leq u < 1;$$

$$(3.25) \quad E(u) = I, \quad 1 \leq u < \infty.$$

The eigenvalues of  $AQ$  are  $\lambda = -1 - 2^{1/2}$ , 0, and  $-1 + 2^{1/2}$ . To the eigenvalue  $\lambda = -1 - 2^{1/2}$  corresponds the projection on the subspace of vectors of the form

$$(3.26) \quad x = \begin{pmatrix} a \\ -a \\ 0 \end{pmatrix},$$

where  $a$  is any complex number. Call this projection  $Q_1$ .  $P_3$  is clearly projection on the subspace corresponding to the eigenvalue  $\lambda = 0$ .  $Q_3$ ,

projection on the subspace of vectors of the form

$$(3.27) \quad x = \begin{pmatrix} a \\ a \\ 0 \end{pmatrix},$$

corresponds to the eigenvalue  $\lambda = -1 + 2^{1/2}$ . Therefore, the resolution of the identity for  $AQ$  is given by

$$(3.28) \quad F(u) = 0, \quad -\infty < u < -1 - 2^{1/2};$$

$$(3.29) \quad F(u) = Q_1, \quad -1 - 2^{1/2} \leq u < 0;$$

$$(3.30) \quad F(u) = Q_1 + P_3, \quad 0 \leq u < -1 + 2^{1/2};$$

$$(3.31) \quad F(u) = I, \quad -1 + 2^{1/2} \leq u < \infty.$$

It is easy to see that  $F(0)H$  neither contains nor is contained in  $E(0)H$ . Hence,  $P_1 \not\geq AQ$  does not hold.

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