MULTIPLIERS ON COMPACT GROUPS

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Abstract. Let a compact group $G$ act continuously both by left and right translation on a Banach space $V$ of integrable functions on $G$. Then $\mathfrak{M}(V)$, the space of bounded linear operators on $V$ commuting with right translation, contains a homomorphic image of $L^1(G)$, whose closure is exactly the set of operators on which $G$ acts continuously. Further, this set is exactly the ideal of compact operators in $\mathfrak{M}(V)$. A restricted version holds for noncompact groups.

1. Compact groups. In this section $G$ denotes a compact group with normalized Haar measure $m$, and the space $L^p(G, m)$, $1 \leq p < \infty$, is briefly denoted by $L^p(G)$. We denote the algebra of finite regular Borel measures on $G$ by $M(G)$.

Let $V$ be a Banach space of functions contained in $L^1(G)$ which is closed under left and right translations.

Definition. We say that $V$ is a $G$-G module if for each $x \in G$, $L(x)f \in V$ and $R(x)f \in V$, and $\|L(x)f - f\|_V \to 0$ and $\|R(x)f - f\|_V \to 0$ as $x \to e$ for each $f \in V$ (the translations $L(x)$ and $R(x)$ are given by $L(x)f(y) = f(xy)$, $R(x)f(y) = f(yx)$, $x, y \in G$, $f \in V$). Furthermore, we require that $\|L(x)f\|_V = \|f\|_V$ and $\|R(x)f\|_V = \|f\|_V$ for each $x \in G$, $f \in V$.

Henceforth $V$ will be a $G$-G module.

As Rieffel [2, p. 447] points out, $V$ is also an $M(G)$-$M(G)$ module, that is, $V$ is closed under left and right convolution by measures.

Now let $\hat{G}$ be the dual of $G$, namely, the set of equivalence classes of continuous unitary irreducible representations of $G$. For $\alpha \in \hat{G}$, let $T_\alpha$ be an element of $\alpha$. Then $T_\alpha$ is a continuous homomorphism of $G$ into $U(n_\alpha)$, the group of $n_\alpha \times n_\alpha$ unitary matrices. Let $\chi_\alpha(x) = \text{Trace}(T_\alpha(x))$, the character of $\alpha$, and let $W_\alpha$ be the linear span of the matrix entry functions of $T_\alpha$. Then $\chi_\alpha$ and $W_\alpha$ depend only on $\alpha$. We call an element in the linear span of $\{W_\alpha : \alpha \in \hat{G}\}$ a trig polynomial.

We note here for later use that $L^1(G)$ has a bounded approximate identity $\{t_s\}$ which is central, that is, $t_s \ast f = f \ast t_s$, $f \in L^1(G)$. This follows since $G$ has a base of invariant neighborhoods of the identity.
Thus $L^1(G)$ has a bounded central approximate identity consisting of trig polynomials.

For $\alpha \in \hat{G}, f \in V$, we have that $f \ast \chi_\alpha \in W_\alpha \cap V$. Since $V$ is left and right invariant, it further holds that $W_\alpha \cap V = W_\alpha$ or $\{0\}$.

**Definition.** Let $\mathcal{M}(V)$ be the space of bounded operators on $V$ which commute with all right translations. Denote the operator norm by $\|\cdot\|_{op}$. For $\mu \in M(G)$, define the operator $j(\mu)$ on $V$ by $j(\mu)f = \mu \ast f$, $f \in V$.

Note that for $T \in \mathcal{M}(V), \mu \in M(G), f \in V$ that $T(f \ast \mu) = (Tf) \ast \mu$.

**Corollary 1.** The map $j$ is a bounded homomorphism of $M(G)$ into $\mathcal{M}(V)$.

Now $\mathcal{M}(V)$ is a right $L^1(G)$-module, and the action is given by $(T \cdot g)f = T(g \ast f)$, for $T \in \mathcal{M}(V), g \in L^1(G), f \in V$. That is, $T \cdot g$ is nothing but $Tj(g)$ (operator composition).

**Definition (Rieffel [2, p. 454]).** The essential part of $\mathcal{M}(V)$, denoted by $\mathcal{M}_e(V)$, is the closed span of $\{T \cdot f : T \in \mathcal{M}(V), f \in L^1(G)\}$. That is, $\mathcal{M}_e(V)$ is just the closed left ideal generated by $jL^1(G)$.

**Theorem 2 (Cohen, Rieffel [2, p. 454]).** The space $\mathcal{M}_e(V) = \mathcal{M}(V)L^1(G)$.

For $x \in G$, let $\delta_x$ be the unit mass at $x$; then for $f \in V$, $\delta_x \ast f = L(x)f$. Now $G$ acts in $\mathcal{M}(V)$ by $T \mapsto Tj(\delta_x)$ for $T \in \mathcal{M}(V)$. Our aim is to characterize those $T \in \mathcal{M}(V)$ for which $\|Tj(\delta_x) - T\|_{op} \to 0$ as $x \to e$. As Rieffel [2, p. 456] observes, these operators are exactly those in the essential part of $\mathcal{M}(V)$.

**Lemma 3.** Let $g$ be a trig polynomial on $G$ and let $T \in \mathcal{M}(V)$. Then $T \cdot g = Tj(g) = j(k)$ for some trig polynomial $k$.

**Proof.** Let $E$ be a finite set contained in $\hat{G}$ which carries $g$, that is $g = \sum_{\alpha \in E} n_\alpha \ast \chi_\alpha$. Thus

$$T \cdot g(f) = T \left( g \ast \left( \sum_{\alpha \in E} n_\alpha \chi_\alpha \ast f \right) \right)$$

$$= T \left( g \ast \left( f \ast \sum_{\alpha \in E} n_\alpha \chi_\alpha \right) \right)$$

$$= (T \cdot g(f)) \ast \sum_{\alpha \in E} n_\alpha \chi_\alpha$$

which is in $V_E$, the span of $\{V \cap W_\alpha : \alpha \in E\}$. Now $V_E$ is a finite
dimensional $G$-$G$ module, and $T \cdot g$ is an operator on $V_E$ which commutes with right translation. Thus there exists a trig polynomial $h$ such that $T \cdot g(f) = h \ast f$ for all $f \in V_E$. But for any $f \in V$,

$$T \cdot g(f) = T \cdot g \left( \sum_{a \in E} n_a \chi_a \ast f \right) = j(h) \left( \sum_{a \in E} n_a \chi_a \ast f \right)$$

$$= j \left( h \ast \sum_{a \in E} n_a \chi_a \right)(f). \quad \square$$

**Theorem 4.** With hypotheses and notation as stated above, $\mathfrak{M}_e(V) = \text{closure}(jL^1(G))$.

**Proof.** If $g \in L^1(G)$, then by the Cohen factorization theorem $g = g_1 \ast g_2, g_1, g_2 \in L^1(G)$. Thus $j(g) = j(g_1)j(g_2) = j(g_1) \cdot g_2 \in \mathfrak{M}(V) \cdot L^1(G)$. (Alternatively, in a not so high-powered fashion, observe directly that $\|j(g)j(\delta_x) - j(g)\|_\text{op} \leq \|g \ast \delta_x - g\|_1 = \|R(x^{-1})g - g\|_1 \to 0$ as $x \to e$. ) Thus $jL^1(G) \subset \mathfrak{M}_e(V)$, a closed set.

Conversely, let $T \in \mathfrak{M}_e(V)$, then $T = S \cdot f$ for some $S \in \mathfrak{M}(V), f \in L^1(G)$. Let $\{t_i\}$ be the bounded central approximate identity consisting of trig polynomials mentioned above. Then

$$\|T - T \cdot t_i\|_{\text{op}} = \|Sj(f) - Sj(f \ast t_i)\|_{\text{op}}$$

$$\leq \|S\|_{\text{op}} \|f - f \ast t_i\|_1 \to 0.$$

By the lemma, $T \cdot t_i \in jL^1(G). \quad \square$

**Theorem 5.** The ideal of compact operators in $\mathfrak{M}(V)$ is equal to $\mathfrak{M}_e(V)$.

**Proof.** By the above, $\mathfrak{M}_e(V) = \text{closure}(jL^1(G))$. If $f \in L^1(G)$ then $\|j(f) - j(f \ast t_i)\|_{\text{op}} \leq \|f - f \ast t_i\|_1 \to 0$. Each $j(f \ast t_i)$ is an operator of finite rank, thus $j(f)$ is compact. The fact that the set of compact operators is norm closed gives containment one way.

Recall the fact that if $\{P_i\}$ is a norm-bounded net of bounded operators on a Banach space $X$ converging strongly to the identity (that is, $P_i x \to x$, each $x \in X$) and if $T$ is a compact operator on $X$, then $\|P_i T - T\|_{\text{op}} \to 0$.

Let $h$ be a central trig polynomial, $T \in \mathfrak{M}(V)$; then $j(h)T = Tj(h)$. In fact, if $f \in V$, then $(j(h)T)(f) = h \ast (Tf) = (Tf) \ast h = T(f \ast h) = T(h \ast f)$. Now let $T$ be a compact operator in $\mathfrak{M}(V)$. We will show that $\|T - T \cdot t_i\|_{\text{op}} \to 0$ and thus $T \in \mathfrak{M}_e(V)$.

Let $f \in V$, then by the Cohen factorization theorem there exist $g \in L^1(G), f_1 \in V$ such that $f = g \ast f_1$. Now
\[ \| j(t) f - f \|_V = \| j(t \ast g)(f_1) - j(g)(f_1) \|_V \]
\[ \leq \| t \ast g - g \|_1 \| f_1 \|_V \xrightarrow{t \to 0} 0, \]

thus \( \{ j(t_i) \} \) converges strongly to the identity in \( \mathfrak{M}(V) \). So
\[ \| T - T \cdot t_i \|_{op} = \| T - T j(t_i) \|_{op} = \| T - j(t_i) T \|_{op} \to 0. \]

**Corollary 6.** For \( T \in \mathfrak{M}(V) \) the following are equivalent:
1. \( \| r(j(\Delta x) - r) \|_{op} \to 0 \) as \( x \to e \),
2. \( T = S \cdot g \), some \( S \in \mathfrak{M}(V) \), \( g \in L^1(G) \),
3. \( FG \text{closure } (G) \),
4. \( T \) is a compact operator.

**Applications.** Let \( 1 < p < \infty \), and \( V = L^p(G) \); then \( \mathfrak{M}(V) \) is the multiplier algebra of \( L^p(G) \). As a particular example, consider \( V = L^2(T) \) (\( T \) is the circle group); then \( \mathfrak{M}(V) \) is identified with \( l^\infty(Z) \), and \( \mathfrak{M}_e(V) \) consists of those bounded sequences \( \{ \phi_n \} \) for which
\[ \sup_n | \phi_n - \phi_n e^{-inz} | \to 0 \text{ as } x \to 0, \]
\( \text{namely } c_0(Z), \text{the sup-norm closure of } L^1(T). \)
For a compact group \( G \) and \( V = L^1(G) \) we get \( \mathfrak{M}(V) = L^\infty(G) \) (see [1]), and \( \mathfrak{M}_e(V) = c_0(G) \). For \( V = C(G) \), \( \mathfrak{M}(V) = M(G) \) and \( \mathfrak{M}_e(V) = L^1(G) \).

2. **Locally compact groups.** Here \( G \) will be a noncompact locally compact group, \( L^1(G) \) the ideal of finite regular Borel measures absolutely continuous with respect to left invariant Haar measure. Theorem 4 does not hold in general in this context. For example, for the real line \( R \), consider \( \mathfrak{M}(L^2(\hat{R})) = L^\infty(R) \), then the essential part is \( L^\infty(R) = L^\infty(R) \cdot C_0(R) \) which is strictly larger than \( C_0(R) = L^1(\hat{R}) \) because it is true that \( jM(G) \cap \mathfrak{M}_e(V) \subset \text{closure } (L^1(G)). \)

We will not require that \( V \) be a space of functions. Here \( V \) will be an isometric left \( G \) module with the action denoted \( xf \) (\( x \in G, f \in V \)), and \( \mathfrak{M}(V) \) will denote the space of bounded operators on \( V \). The map \( j: M(G) \to \mathfrak{M}(V) \), given by \( j(\mu)(f) = \int_0 (xf) \, d\mu(x), f \in V, \mu \in M(G), \) is a homomorphism with \( \| j(\mu) \|_{op} \leq \| \mu \| \). The essential part of \( \mathfrak{M}(V) \), denoted by \( \mathfrak{M}_e(V) \), equals \( \mathfrak{M}(V)(jL^1(G)). \)

The following holds for \( T \in \mathfrak{M}(V) : T \in \mathfrak{M}_e(V) \) if and only if
\[ \| T j(\Delta x) - T \|_{op} \to 0 \text{ as } x \to e. \]

**Theorem 7.** \( jM(G) \cap \mathfrak{M}_e(V) = jM(G) \cap \text{closure } (jL^1(G)). \)

**Proof.** As before it is clear that \( \text{closure } (jL^1(G)) \subset \mathfrak{M}_e(V). \) Now let \( \mu \in M(G) \) such that \( j(\mu) \in \mathfrak{M}_e(V) \); then there exist \( T \in \mathfrak{M}(V), g \in L^1(G) \) such that \( j(\mu) = T \cdot g. \) Let \( \{ u_i \} \) be an approximate identity in \( L^1(G) \), then \( \mu \ast u_i \in L^1(G) \) for each \( i \) and...
\[ \| j(\mu) - j(\mu * u_i) \|_{op} = \| T \cdot g - (T \cdot g)(j(u_i)) \|_{op} \]
\[ = \| Tj(g) - Tj(g)j(u_i) \|_{op} \]
\[ \leq \| T \|_{op} \| g - g * u_i \|_1 \to 0. \]

**Corollary 8.** Let \( \mu \in M(G) \), then the following are equivalent:

1. \( \| j(\mu * \delta_x) - j(\mu) \|_{op} \to 0 \) as \( x \to e \),
2. \( j(\mu) \in \text{closure}(jL^1(G)) \).

**Application.** For \( 1 < p < \infty \), let \( L^p(G) \) be the \( L^p \) space of left invariant Haar measure. Corollary 8 characterizes the measures which can be approximated in the \( L^p \)-operator norm by \( L^1(G) \). Let \( V \) be the direct sum of all (classes of) irreducible unitary continuous representations of \( G \); then the \( V \)-operator norm is the \( C^* \) norm \( \| \cdot \|_\delta \) of \( L^1(G) \) and \( M(G) \). Thus we have another proof of our characterization of \( M_0(G) \), the measures approximable in \( \| \cdot \|_\delta \) by \( L^1(G) \) (see [1]).

**Bibliography**


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