COMMUTING OPERATOR SOLUTIONS OF ALGEBRAIC EQUATIONS

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Abstract. Let $G(w, z)$ be a complex polynomial, and $S$ a bounded operator of scalar type on a complex Banach space, whose spectrum avoids the points $\lambda$ for which $G(\lambda, z) = 0$ has multiple roots $z$. The form of a bounded operator $T$ which commutes with $S$ and satisfies $G(S, T) = 0$ is established.

1. Introduction. Fix a Banach space $X$ over the complex numbers $\mathbb{C}$, and let $\mathcal{B}$ denote the Banach algebra of all bounded linear operators on $X$. Given $S \in \mathcal{B}$ of scalar type, and given a polynomial in two indeterminates

$$G(w, z) = a_n(w)z^n + \cdots + a_1(w)z + a_0(w) \quad (a_i(w) \in \mathbb{C}[w]),$$

we seek operators $T \in \mathcal{B}$ such that

$$(E) \quad T \text{ commutes with } S \text{ and } G(S, T) = 0.$$ 

Denoting the spectrum of $S$ by $\sigma$, we assume:

$$\text{(A) For each } \lambda \in \sigma, \text{ the polynomial } G(\lambda, z) \text{ has } n \text{ distinct complex roots } t_1(\lambda), \cdots, t_n(\lambda).$$

We shall establish:

Theorem. Notation and assumptions as above, there exist $T_1, \cdots, T_n \in \mathcal{B}$ satisfying (E), and having the following property: $T \in \mathcal{B}$ satisfies (E) if and only if there exist $F_1, \cdots, F_n \in \mathcal{B}$ such that

$$F_i \text{ commutes with } S,$$

$$F_iF_j = 0 \quad \text{for } i \neq j, \quad \sum F_i = I,$$

$$T = \sum T_iF_i.$$

An operator $S$ is of scalar type [1, p. 332] if $S$ admits a resolution of the identity $E(\cdot)$, and if moreover $S$ can be recovered from $E(\cdot)$ by integration over $\sigma$:

$$S = \int \lambda E(d\lambda).$$

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In particular, a normal operator on a Hilbert space is of scalar type. In the theorem, $G$ need not be irreducible, but by (A) no repeated factors are permitted in the prime decomposition of $G$. The idempotents $F_i$ are not asserted to be values of $E(\cdot)$, and indeed need not be, if $S$ has multiplicity greater than 1.

Foguel [2] proved this theorem for the special case $G(w, z) = g(z) - w$, $g$ a complex polynomial, and we imitate his proof. For a given solution $T$ of (E), the main step in constructing the $F_i$ is to check that $G_2(S, T)$ is invertible, where $G_2(w, z) = \partial G(w, z)/\partial z$. In Foguel's case, $G_2(w, z) = g'(z)$ is independent of $w$, and the existence of $g'(T)^{-1}$ follows immediately from (A) and the spectral mapping theorem. The general proof below uses maximal ideals, and we are indebted to the referee for a substantial simplification of our original argument.

2. Proof of the theorem. Let $\mathfrak{B}$ denote the Banach algebra of all essentially bounded measurable complex functions on $\sigma$.

**Lemma 1.** There exist $t_1, \ldots, t_n \in \mathfrak{B}$ such that, for each $\lambda \in \sigma$,

1. $G(\lambda, z) = a_n(\lambda) \prod_j (z - t_j(\lambda))$,
2. $G_2(\lambda, z) = a_n(\lambda) \Sigma_i \prod_{j \neq i} (z - t_j(\lambda))$.

**Proof.** Let $K$ be an oriented cut in $\mathbb{C}$ joining the singularities $\lambda_1, \ldots, \lambda_k$ of the algebraic function(s) determined by $G$. Then the roots $t_i(\lambda)$ can be chosen to be holomorphic in $\mathbb{C} - K$. If each $t_i$ is analytically continued to $K' = \mathbb{C} - \{\lambda_1, \ldots, \lambda_k\}$ from the left, say, then the extended $t_i$ are defined and locally bounded on $\mathbb{C}' = \mathbb{C} - \{\lambda_1, \ldots, \lambda_k\}$, have, at worst, jump discontinuities from the right along $K'$, and have for values $\{t_i(\lambda)\}$ precisely the set of roots of $G(\lambda, z)$, for each $\lambda \in \mathbb{C}'$. Assumption (A) provides that $\sigma \subset \mathbb{C}'$, and in particular that $a_n(\lambda) \neq 0$ for $\lambda \in \sigma$; hence restricting the $t_i$ to $\sigma$ establishes (1), from which (2) is immediate.

By [1, Lemma 6, p. 341], the map $\mathfrak{M} \to \mathfrak{B}$ given by $f \mapsto f(S) = \int f(\lambda) E(d\lambda)$ is a continuous algebra homomorphism. Clearly it extends by $z \mapsto z$ to a homomorphism $\mathfrak{M}[z] \to \mathfrak{B}[z]$, which carries (1) and (2) over to relations

3. $G(S, z) = a_n(S) \prod_j (z - T_j)$,
4. $G_2(S, z) = a_n(S) \Sigma_i \prod_{j \neq i} (z - T_j)$,

in which we have set $T_j = t_j(S)$. Then each $T_j$ commutes with $S$ and, by (3), obeys $G(S, T_j) = 0$.

Now suppose that $T$ satisfies (E). Then in particular $T$ must commute with $E(\cdot)$ [1, Theorem 5, p. 329] and hence with the $T_j$, so that $z \mapsto T$ defines a homomorphism $\mathfrak{B}[z] \to \mathfrak{B}$, which carries (3) and (4) to
(5) \[ 0 = G(S, T) = a_n(S)\Pi_i(T - T_i), \]
(6) \[ G_2(S, T) = a_n(S)\Sigma_j\Pi_{i \neq j}(T - T_j). \]

**Lemma 2.** \( G_2(S, T)^{-1} \) exists in \( \mathfrak{g} \) and commutes with \( S \) and \( T \).

**Proof.** Let \( \mu: \mathfrak{g} \to \mathbb{C} \) denote any nonzero continuous homomorphism, where \( \mathfrak{g} \subset \mathfrak{g} \) is the (commutative) full algebra generated by \( E(\cdot) \) and \( T \) [1, p. 342]. Then \( \lambda = \mu(S) \in \sigma \), for otherwise \( (S - \lambda I)^{-1} \in \mathfrak{g} \) by definition of “full algebra”, and \( I = (S - \lambda I)(S - \lambda I)^{-1} \) would go by \( \mu \) to \( 1 = 0 \). Hence \( G(\lambda, \mu(T)) = \mu(G(S, T)) = 0 \), and then \( \mu(G_2(S, T)) = G_2(\lambda, \mu(T)) \neq 0 \), by (A). Thus \( G_2(S, T) \) lies in no maximal ideal of \( \mathfrak{g} \), so is invertible in \( \mathfrak{g} \).

Now for each \( i \), set

(7) \[ F_i = G_2(S, T)^{-1}a_n(S)\Pi_{j \neq i}(T - T_j). \]

To verify (D), notice that \( \Sigma_i F_i = I \) follows from (6). For \( i \neq j \), \( F_i F_j \) contains each factor \( T - T_k \) at least once, so vanishes by (5); and then \( F_i^2 = F_i(\Sigma_j F_j) = F_i \) follows. Similarly (7) and (5) give \( (T - T_i) F_i = 0 \) for each \( i \), and summing yields \( T = \Sigma_i T_i F_i \), to conclude the “only if” part of the proof.

Conversely, suppose that \( F_1, \ldots, F_n \) obey (D). Then each \( F_i \) commutes with \( E(\cdot) \), hence with each \( T_j \), and \( T = \Sigma_i T_i F_i \) commutes with \( S \). Moreover, it follows by induction that \( T^k F_i = T_i^k F_i \), hence that \( G(S, T) F_i = 0 \), for each \( i \). Summing yields \( G(S, T) = 0 \), to conclude the “if” part.

**References**


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