ON THE EXISTENCE OF DOUBLE SINGULAR INTEGRALS FOR KERNELS WITHOUT SMOOTHNESS

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Abstract. Calderón and Zygmund have proved the pointwise convergence of singular integrals in $\mathbb{R}^n$ for locally integrable homogeneous kernels whose even part is locally in $L \log L$ by change to polar coordinates and use of the boundedness in $L^p$ of the maximal operator of the one-dimensional Hilbert transformation. The present note shows how analogous results for double singular integrals can be derived from boundedness of the maximal operator of the double Hilbert transform.

For $i = 1, 2$ let $K_i$ be a complex valued function defined in $\mathbb{R}^n$ which is (positively) homogeneous of degree $-n_i$, i.e., $K_i(\lambda x_i) = \lambda^{-n_i} K_i(x_i)$ for $x_i \neq 0, \lambda > 0$, locally integrable away from the origin, of mean value zero on the unit sphere of $\mathbb{R}^n$, i.e.,

$$\int_{|x'_i| = 1} K_i(x'_i) \, dx'_i = 0$$

(where $dx'_i$ denotes ordinary surface measure on $S^{n_i-1} = \{ x'_i : |x'_i| = 1 \}$) and whose even part belongs to $L \log L$ on the unit sphere, i.e.,

$$\int_{|x'_i| = 1} \left| K_i(x'_i) + K_i(-x'_i) \right| \log^+ \left| K_i(x'_i) + K_i(-x'_i) \right| \, dx'_i < \infty.$$

A. Zygmund called attention to the problem of showing by the methods of [2] that if

$$f^*(x) = \sup \{ \left| \tilde{f}_{\varepsilon_1, \varepsilon_2}(x) \right| : \varepsilon_1, \varepsilon_2 > 0 \}$$

where

$$\tilde{f}_{\varepsilon_1, \varepsilon_2}(x_1, x_2) = \int_{|x_1 - y_1| > \varepsilon_1} \int_{|x_2 - y_2| > \varepsilon_2} K_1(x_1 - y_1) K_2(x_2 - y_2) f(y_1, y_2) \, dy_2 \, dy_1$$

then

$$\|f^*\|_p \leq A_p \|f\|_p$$

for $1 < p < \infty$.

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$K_1, K_2$ restricted to $S^{n-1}, S^{n-1}$ satisfy the Dini condition $\int_0^1 t^{-1}\omega_i(t)dt < \infty$ for $i = 1, 2$, this was shown by Cotlar in [3]. The purpose of this note is to prove the following

**Proposition.** Suppose $K_1, K_2$ are homogeneous of degree $-n_1, -n_2$, respectively, locally integrable and of mean value zero on $S^{n-1}$ and satisfy (1), then for $f^*$ defined by (2), (3) is valid. Moreover if $v$ indicates how many of $K_1, K_2$ are odd ($v = 0, 1, 2$) then $A_p = O((p-1)^{-v})$ as $p \downarrow 1$ ($O(p^{v+})$ as $p \rightarrow \infty$).

The proof requires the following

**Lemma.** Let $f \in L^p(R^2)$ and

$$f(\xi_1, \xi_2) = \sup_{\eta_1, \eta_2 > 0} \pi^{-2} \int_{|\xi_1-\tau_1| > \eta_1} \int_{|\xi_2 - \tau_2| > \eta_2} (\xi_1 - \tau_1)^{-1}(\xi_2 - \tau_2)^{-1}f(\tau_1, \tau_2)d\tau_2d\tau_1$$

then $\|f\|_p \leq A_p \|f\|_p$ where $A_p = O((p-1)^{-2})$ for $p \downarrow 1$.

For the maximal double conjugate function of a periodic function the analogous assertion follows from the arguments of [6, especially pp. 228-233] and with $A_p = ((p-1)^{-4})$ is Theorem 3 of [4]. Again with $A_p = O((p-1)^{-4})$ the lemma is contained in [3, Theorem 3, p. 102]. A proof analogous to that of Theorem 6' of [6] might run briefly as follows.

Let

$$P(\xi, \eta) = \pi^{-1}(\xi^2 + \eta^2)^{-1}, \quad Q(\xi, \eta) = \pi^{-1}(\xi^2 + \eta^2)^{-1}$$

then $(i\pi)^{-1}(\xi - i\eta)^{-1} = P(\xi, \eta) - iQ(\xi, \eta)$, hence

$$[P(\cdot, \eta_1) \otimes P(\cdot, \eta_2) - Q(\cdot, \eta_1) \otimes Q(\cdot, \eta_2)] * f(\xi_1, \xi_2)$$

and

$$-[P(\cdot, \eta_1) \otimes Q(\cdot, \eta_2) + Q(\cdot, \eta_1) \otimes P(\cdot, \eta_2)] * f(\xi_1, \xi_2)$$

are the real and imaginary parts, respectively, of

$$F(\xi_1, \xi_2) = (i\pi)^{-2} \int \int f(\tau_1, \tau_2)(\xi_1 - \tau_1)^{-1}(\xi_2 - \tau_2)^{-1}d\tau_1d\tau_2$$

($\xi_j = \xi_j + i\eta_j$).

It will be seen that $F$ is in $H^p$. In what follows $C$ will denote a constant not necessarily the same at each occurrence. It is well known that, e.g.,
\[ \| Q(\cdot, \eta_1) \ast f(\cdot, \xi_2) \|_p \leq C(p, p') \| f(\cdot, \xi_2) \|_p \quad (p^{-1} + p'^{-1} = 1) \]

hence \[ | F(\xi_1 + i\eta_1, \xi_2 + i\eta_2) | : \eta_1, \eta_2 > 0 \]
then \[ \| F^* \|_p \leq C(p, p') \| f \|_p \]
Hence consideration of the real part of \( F \) leads to
\[ \| \sup \{ P(\cdot, \eta_1) \otimes P(\cdot, \eta_2) \ast f \} : \eta_1, \eta_2 > 0 \|_p \leq C(p, p') \| f \|_p \]
It remains to observe that if \( H(\xi, \eta) = (\pi \xi)^{-1}(1 - \chi(-\eta, \eta)), \chi(-\eta, \eta) \) being the characteristic function of the interval \((-\eta, \eta), \)
\[ H(\cdot, \eta_1) \otimes H(\cdot, \eta_2) - Q(\cdot, \eta_1) \otimes Q(\cdot, \eta_2) \]
and \[ | H(\xi, \eta) - Q(\xi, \eta) | \leq \eta^{-1} \psi(\eta^{-1}) \] where \( \psi \) is even, nonincreasing in \((0, \infty)\) and integrable so that, e.g.,
\[ \| \sup_{\eta_1 > 0} \eta^{-1} \psi(\eta^{-1}) \ast g(\cdot, \xi_2) \|_p \leq C(p, p') \| g(\cdot, \xi_2) \|_p \]
(e.g., by Lemma 1 of Chapter II of [1] where
\[ g(\xi_1, \xi_2) = \sup_{\eta_2 > 0} | H(\cdot, \eta_2) \ast f(\xi_1, \cdot)(\xi_2) | . \]
Proof of the Proposition. First of all, for a.e. \((x_1, x_2)\) and any \( \epsilon_1, \epsilon_2 > 0, \)
\[ \int_{|x_1| > \epsilon_1} \int_{|x_2| > \epsilon_2} | K_1(y_1)K_2(y_2)f(x_1 - y_1, x_2 - y_2) | \, dy_2 dy_1 < \infty . \]
This follows as in [2, p. 292] by integration of the last integral over any compact subset of \( R^n \times R^n \). In fact, let \( B_i = \{ x_i : x_i \in R^n, \, |x_i| \leq r_i \} \) then the integral of the left-hand side of (4) over \( B_1 \times B_2 \) is at most
\[ \int_{|y_1'| = 1} \int_{|y_2'| = 1} | K_1(y_1')K_2(y_2') | \int_{B_1} \int_{B_2} \int_{t_1}^{\infty} \int_{t_2}^{\infty} | f(x_1 - y_1' t_1, x_2 - y_2' t_2) | \, d_1 \, d_2 \, dx_1 dx_2 \, dy_1' dy_2' \]
\[ \leq C \epsilon_1 \epsilon_2 \int_{r_1}^{1/p - 1/p} \int_{r_2}^{n_1/p + 1/p} \int_{r_2}^{n_2/p + 1/p} \| f \|_p . \]
If $K_1, K_2$ are both odd then
\[
\tilde{f}_{\epsilon_1, \epsilon_2}(x_1, x_2) = -\int_{|y_1|>\epsilon_1} \int_{|y_2|>\epsilon_2} K_1(y_1) K_2(y_2) f(x_1 - y_1, x_2 + y_2) dy_2 dy_1
\]
\[
= -\int_{|y_1|>\epsilon_1} \int_{|y_2|>\epsilon_2} K_1(y_1) K_2(y_2) f(x_1 + y_1, x_2 - y_2) dy_2 dy_1
\]
\[
= \int_{|y_1|>\epsilon_1} \int_{|y_2|>\epsilon_2} K_1(y_1) K_2(y_2) f(x_1 + y_1, x_2 + y_2) dy_2 dy_1.
\]
Hence by (4)
\[
\tilde{f}_{\epsilon_1, \epsilon_2}(x_1, x_2)
\]
\[
= (1/4) \int_{|y'_1|=1} \int_{|y'_2|=1} K_1(y'_1) K_2(y'_2) \tilde{f}_{\epsilon_1, \epsilon_2}(x_1, x_2; y'_1, y'_2) dy'_2 dy'_1
\]
where
\[
\tilde{f}_{\epsilon_1, \epsilon_2}(x_1, x_2; y'_1, y'_2) = \int_{|t_1|>\epsilon_1} \int_{|t_2|>\epsilon_2} f(x_1 - y'_1 t_1, x_2 - y'_2 t_2) t_1^{-1} t_2^{-1} dt_1 dt_2.
\]
Let
\[
f(x_1, x_2; y'_1, y'_2) = \sup_{\epsilon_1, \epsilon_2 > 0} |\tilde{f}_{\epsilon_1, \epsilon_2}(x_1, x_2; y'_1, y'_2)|.
\]
$f(\cdot, \cdot; y'_1, y'_2)$ restricted to any plane parallel to $y'_1$ and $y'_2$ is the maximal function of the truncated ordinary double Hilbert transforms of $f$ restricted to such planes. Consequently by the lemma
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1 - y'_1 t_1, x_2 - y'_2 t_2; y'_1, y'_2)^p dt_2 dt_1
\]
\[
\leq A_p \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x_1 - y'_1 t_1, x_2 - y'_2 t_2)|^p dt_2 dt_1
\]
and integration of this inequality over the space of planes parallel to $y'_1$ and $y'_2$ gives $\|f(\cdot; y'_1, y'_2)\|_p \leq A_p \|f\|_p$. (2) now follows from (5) and Minkowski's inequality for integrals as in [2].

If $K_1, K_2$ are not both odd functions it appears sufficient to consider the case when both are even; if one is odd and the other even the following argument simplifies in an obvious manner. If as in [2, p. 299] $\phi$ denotes a continuously differentiable function of the real variable $t$, $t \geq 0$, equal to zero in $(0, \frac{1}{4})$ and to 1 in $(\frac{1}{4}, \infty)$ then
\[ \tilde{f}_{x_1,x_2}(x_1,x_2) = \left( \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} - \int_{\mathbb{R}^{n_1}} \int_{|x_2-y_2|<\varepsilon_2} \right. \\
\cdot \int_{|x_1-y_1|<\varepsilon_1} \int_{\mathbb{R}^{n_2}} + \int_{|x_1-y_1|<\varepsilon_1} \int_{|x_2-y_2|<\varepsilon_2} \\
\left. \cdot K_1(x_1-y_1)\phi(|x_1-y_1|\varepsilon_1^{-1}) \\
\cdot K_2(x_2-y_2)\phi(|x_2-y_2|\varepsilon_2^{-1})f(y_1,y_2)dy_2dy_1. \right) \tag{6} \]

The integrand is integrable in \( R^{n_1+n_2} \) for a.e. \((x_1, x_2)\) by (4). Let \( R \) denote the (vector valued) Riesz kernel in \( R^{n_1} \) or \( R^{n_2} \) according to the context and define \((n_1 \times 1), (1 \times n_2)\) and \((n_1 \times n_2)\) vector valued functions

\[ g_{10}(x_1, x_2) = - \text{p.v.} R * f(\cdot, x_2)(x_1), \]
\[ g_{01}(x_1, x_2) = - \text{p.v.} R * f(x_1, \cdot)(x_2), \]
\[ g_{11}(x_1, x_2) = \text{p.v.} (R \otimes R) * f(x_1, x_2). \]

According to the lemma on pp. 299–300 of [2] if \( K_{i1} = \text{p.v.} R * K_{i1}, K_{i2} = \text{p.v.} R * (K \phi(| \cdot |)) \) then \( K_{i1}, K_{i2} \) are odd, \( K_{i1} \) is homogeneous of degree \(-n_i\), for \(|x_i| \geq 1\)

\[ |K_{i1}(x_i) - K_{i2}(x_i)| \leq C \int_{|y_i'|=1} |K_i(y_i')|dy_i'|x_i|^{-n_i-1} \]

and there are functions \( G_i \) homogeneous of degree 0 such that for \(|x_i| \leq 1, |K_{i2}| \leq G_i \) and \( \int_{|x_i'|=1} G_i(x_i')dx_i' < \infty \). Then by (5.10) of [2] the first integral in (6) equals

\[ \varepsilon_1^{-n_1} \int K_1(x_1-y_1)\phi(|x_1-y_1|\varepsilon_1^{-1}) \\
\cdot \int K_2((x_2-y_2)\varepsilon_2^{-1})g_{01}(y_1,y_2)dy_2dy_1 \tag{7} \]

and by §5 of [2] as a function of \((y_1, x_2)\) the inner integral is in \( L^p \), hence for a.e. \( x_2^0 \subseteq \mathbb{R}^{n_2} \) the restriction to \( x_2 = x_2^0 \) is in \( L^p(\mathbb{R}^{n_2}) \) and hence again by (5.10) of [2] (7) equals

\[ \varepsilon_1^{-n_1}\varepsilon_2^{-n_2} \int \int [K_{12}((x_1-y_1)\varepsilon_1^{-1}) \otimes K_{22}((x_2-y_2)\varepsilon_2^{-1})]g_{11}(y_1,y_2)dy_2dy_1. \]

A similar procedure with the second and third terms on the right-hand side of (6) leads to
\[
\left| \int_{\varepsilon_1, \varepsilon_2} \varepsilon_1^n \varepsilon_2 \right| \int_{|x_1 - y_1| < \varepsilon_1} \int_{|x_2 - y_2| < \varepsilon_2} (K_{11}(x_1 - y_1) \otimes K_{21}(x_2 - y_2)) \cdot g_{11}(y_1, y_2) dy_1 dy_2
\]

\[+ \varepsilon_2 \left( \int_{|x_1 - y_1| < \varepsilon_1} G_1 \left( \frac{x_1 - y_1}{|x_1 - y_1|} \right) + C \int_{|x_1 - y_1| > \varepsilon_1} (x_1 - y_1)^{-1} e_1^{n_1-1} \right)
\]

\[\cdot \left| \int_{|x_2 - y_2| < \varepsilon_2} K_{21}(x_2 - y_2) \cdot g_{11}(y_1, y_2) dy_2 dy_1 \right|
\]

\[+ \int_{|x_1 - y_1| < \varepsilon_1} \int_{|x_2 - y_2| < \varepsilon_2} G_1 \left( \frac{x_1 - y_1}{|x_1 - y_1|} \right) (x_2 - y_2) e_2^{-1} |e_2^{n_2-1} \cdot g_{11}(y_1, y_2) dy_2 dy_1
\]

\[+ (3 \text{ similar terms obtained by interchanging } x_1, y_1, e_1, G_1, n_1 \text{ with } x_2, y_2, e_2, G_2, n_2 \text{ in the preceding 3 integrals})
\]
\[ + \int_{|x_1-y_1|<\varepsilon_1} \int_{|x_2-y_2|<\varepsilon_2} G_1\left(\frac{x_1 - y_1}{x_1 - y_1}\right) G_2\left(\frac{x_2 - y_2}{x_2 - y_2}\right) \cdot |g_{11}(y_1, y_2)| \, dy_2 \, dy_1 \\
+ C \int_{|x_1-y_1|>\varepsilon_1} \int_{|x_2-y_2|>\varepsilon_2} |(x_1 - y_1)^{1-\delta_1}|^{-n_1-1} |(x_2 - y_2)^{1-\delta_2}|^{-n_2-1} \cdot |g_{11}(y_1, y_2)| \, dy_2 \, dy_1. \]

Substitution in the estimate for \( f_{n_1, n_2}(x_1, x_2) \), the result for products of odd kernels, Theorems 1 and 6 of [2],
\[ ||g_{10}||_p, ||g_{01}||_p \leq C_{p'} ||f||_p, \quad ||g_{11}||_p \leq C (p')^2 ||f||_p \]
and the fact that the "outer" operators (in all but the first term) are positive imply (3).

**Remark.** Completely analogously it can be shown by induction that if \( K_i \in L_{\text{loc}}(R^{n_i} - \{0\}), 1 \leq i \leq N \), are several kernels all satisfying the conditions of the proposition then
\[ f^*(x) = \sup_{\varepsilon_i > 0} \left| \int_{|x_1-y_1|>\varepsilon_1} \cdots \int_{|x_N-y_N|>\varepsilon_N} (K_1 \otimes \cdots \otimes K_N)(x - y) \cdot f(y) dy_1, \cdots, dy_N \right| \]
satisfies (3), where \( x = (x_1, \ldots, x_N) \in R^{n_1 + \cdots + n_N} \).

It is also clear that analogous results hold for products of several kernels of any of the types discussed in [2].

**References**


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