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SEMIGROUPS ON ACYCLIC PLANE CONTINUA

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ABSTRACT. It is shown that an acyclic irreducible plane continuum which admits the structure of a topological semigroup is an arc if it has an identity, and is either an arc, is trivial, or is decomposible into an arc if it satisfies $M^2=M$. This extends some results of Friedberg and Mahavier concerning semigroups on chainable continua.

Let M be a topological semigroup with minimal ideal K whose underlying space is a nondegenerate compact metric continuum. If M has an identity, M is called a clan.

Under the assumption that M is chainable, Friedberg and Mahavier [3] showed that if M is a clan it is an arc, and if $M^2=M$ then either M is trivial, M is an arc, or $M|K$ is an arc and M is irreducible from a one-sided identity to some point. In this note we extend these results (using essentially the same arguments) by replacing the condition that M be chainable by the condition that M be an acyclic (i.e., contains no simple closed curve) plane continuum which is irreducible between two points. (Every nondegenerate chainable continuum is homeomorphic to such a continuum.)

THEOREM 1. *If M is an acyclic clan in the plane, then M is arcwise connected.*

PROOF. Let G be a closed subgroup of M with identity e and let $C(e)$ be the component of G containing e. $C(e)$ is a subcontinuum of M and is a group. Suppose $C(e)$ is nondegenerate. Then it is homogeneous and by [4] contains an arc; so by [1] it is a simple closed curve, contradicting the assumption that M is acyclic. Thus $C(e)$ is degenerate and G is totally disconnected. Then M is arcwise connected by [6].

COROLLARY. *If M is an acyclic plane continuum which is irreducible between two of its points it is an arc.*

REMARK. The referee has observed that except for the existence of the one-sided identity, the conclusion of the next theorem follows from Hunter's argument in [5, Theorem 8], without the assumption

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that M be acyclic. Also, a simplification suggested by the referee has been employed in the next argument.

THEOREM 2. *If M is an acyclic plane continuum which is irreducible between two of its points and $M^2 = M$, then either*

(1) *$M = K$ and the multiplication on K is trivial,*

(2) *M is an arc, or*

(3) *M has a one-sided identity e, $M | K$ is an arc, and M is irreducible from e to some point.*

PROOF. Let E denote the set of idempotent elements of M, and for e in E, let H_e be the maximal subgroup containing e. Since M is acyclic, K is not the cartesian product of two nondegenerate continua [**5**, Lemma 2, p. 238]; so K is a group or multiplication in K is trivial [**7**, Corollary 1]. As in the proof of Theorem 1, if K is a group it is degenerate. In either case multiplication in K is trivial and K is a subset of E.

Now assume that $M \neq K$ and M is not an arc. Suppose M has no one-sided identity. Since M is irreducible between two points a and b, there exist points e and f in $E \backslash K$ such that $a \in H_e$, $b \in H_f$, H_e and H_f are connected, and $M = (eMe) \cup (fMf)$ [**7**, Theorem 5]. But H_e and H_f are degenerate so M is irreducible from e to f. Since eMe and fMf are acyclic plane clans, they are arcwise connected by Theorem 1. Then M is an arc from e to f, a contradiction. Thus M has a right (or left) identity e.

Then $Me = M$ and $eM = eMe$ is either degenerate or arcwise connected. If eM is degenerate, $e \in K$ and $Me = M = K$, a contradiction. Hence $eM = eMe$ is a nondegenerate arcwise connected clan with e as its identity. Let T be an arc in eM from e to its minimal ideal K' such that $T \cap K'$ is degenerate. Clearly $K' \subseteq K$. Since each of aT and bT is a continuous image of T, each is either degenerate or arcwise connected, and there is an α and a β such that each of α and β is an arc or degenerate, $\alpha \subseteq aT$, $\beta \subseteq bT$, α contains a, β contains b and each of α and β intersects K at only one point. Since M is irreducible from a to b, $M = \alpha \cup K \cup \beta$. If both a and b belong to K, $K = M$, so let $e \in \beta \backslash K$. If $e \neq b$, e possesses a euclidean (1-dimensional) neighborhood and since e is a right identity, $e \in K$, a contradiction [**2**, Lemma 4]. Hence $e = b$ and (3) holds.

REMARK. An application of Theorem 1 to some nonchainable continua would be as follows: no continuum in the plane consisting of an infinite half-ray "spiraling down" upon a nondegenerate acyclic continuum admits the structure of a topological semigroup with identity.

References

1. R. H. Bing, *A simple closed curve is the only homogeneous bounded plane continuum that contains an arc*, Canad. J. Math. **12** (1960), 209–230. MR **22** #1869.

2. Haskell Cohen and R. J. Koch, *Acyclic semigroups and multiplications on two-manifolds*, Trans. Amer. Math. Soc. **118** (1965), 420–427. MR **30** #5283.

3. M. Friedberg and W. S. Mahavier, *Semigroups on chainable and circle-like continua*, Math. Z. **106** (1968), 159–161. MR **38** #263.

4. A. M. Gleason, *Arcs in locally compact groups*, Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 663–667. MR **12**, 391.

5. R. P. Hunter, *Note on arcs in semigroups*, Fund. Math. **49** (1960/61), 233–245. MR **23** #A628.

6. R. J. Koch, *Threads in compact semigroups*, Math. Z. **86** (1964), 312–316. MR **30** #1499.

7. R. J. Koch and A. D. Wallace, *Admissibility of semigroup structures on continua*, Trans. Amer. Math. Soc. **88** (1958), 227–287. MR **20** #1729.

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