PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, Number 2, May 1971

ON SOME PRODUCTS INVOLVING PRIMES

S. UCHIYAMA

ABSTRACT. Asymptotic formulae are given for the products $P_l(x)$ ($l=1,3$) defined below.

We put, for $x \geqq 2$ and $l=1$ and 3,

$$P_l(x) = \prod_{p \leq x;\, p \equiv l(4)} \left(1 - \frac{1}{p}\right),$$

where the product is taken over the specified primes p. Our aim in the present note is to show that

$$\begin{aligned} P_1(x) &= (\pi A_1 e^{-C})^{1/2} (\log x)^{-1/2} + O((\log x)^{-3/2}), \\ P_3(x) &= \left(\frac{\pi A_3 e^{-C}}{2}\right)^{1/2} (\log x)^{-1/2} + O((\log x)^{-3/2}), \end{aligned} \tag{1}$$

where C denotes the Euler constant and

$$A_l = \prod_{p \equiv l(4)} \left(1 - \frac{1}{p^2}\right)$$

(so that $A_1 A_3 = 8/\pi^2$).

Now, let us define $\chi(n)=0$ for even n and $=(-1)^{(n-1)/2}$ for odd n. Then, $\chi(n)$ is a residue character (mod 4), and the corresponding L-series $L(s,\chi)=\sum_{n=1}^{\infty} \chi(n) n^{-s}$ represents a continuous function of s for $s>0$. In particular, we have $L(1,\chi)=\pi/4$ and

$$L(1,\chi) = \prod_{p \leq x} \left(1 - \frac{\chi(p)}{p}\right)^{-1} + O\left(\frac{1}{\log x}\right)$$

(cf. [1, §109]), whence

$$\frac{P_3(x)}{P_1(x)} = \frac{\pi}{4} A_3 + O\left(\frac{1}{\log x}\right). \tag{2}$$

On the other hand, we have by a well-known theorem due to F. Mertens (cf. [1, §36])

$$P_1(x) P_3(x) = \frac{2e^{-C}}{\log x} + O\left(\frac{1}{\log^2 x}\right). \tag{3}$$

Received by the editors October 22, 1970.
AMS 1970 subject classifications. Primary 10H25.
Key words and phrases. Dirichlet L-series, Mertens' theorem.

The result (1) follows at once from (2) and (3).

We note that our asymptotic formula for $P_1(x)$ will give a solution to a problem recently posed by D. Suryanarayana in [2].

References

1. E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*. Vols. 1, 2, 2nd ed., Chelsea Publishing Co., New York, 1953. MR **16**, 904.

2. D. Suryanarayana, *Problem* 17 (1), Bull. Amer. Math. Soc. **76** (1970), 976.

Shinshû University, Matsumoto, Japan