ON SOME PRODUCTS INVOLVING PRIMES

S. UCHIYAMA

Abstract. Asymptotic formulae are given for the products $P_l(x)$ ($l = 1, 3$) defined below.

We put, for $x \geq 2$ and $l = 1$ and $3$,

$$P_l(x) = \prod_{p \leq x; p \equiv l (4)} \left(1 - \frac{1}{p}\right),$$

where the product is taken over the specified primes $p$. Our aim in the present note is to show that

$$P_1(x) = \left(\pi A_1 e^{-C}\right)^{1/2} (\log x)^{-1/2} + O((\log x)^{-3/2}),$$

and

$$P_3(x) = \left(\pi A_3 e^{-C}\right)^{1/2} \left(\frac{2}{1/2}\right) (\log x)^{-1/2} + O((\log x)^{-3/2}),$$

where $C$ denotes the Euler constant and

$$A_l = \prod_{p \equiv l (4)} \left(1 - \frac{1}{p^2}\right)$$

(so that $A_1 A_3 = 8/\pi^2$).

Now, let us define $\chi(n) = 0$ for even $n$ and $\chi(n) = (-1)^{(n-1)/2}$ for odd $n$. Then, $\chi(n)$ is a residue character (mod 4), and the corresponding $L$-series $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s}$ represents a continuous function of $s$ for $s > 0$. In particular, we have $L(1, \chi) = \pi/4$ and

$$L(1, \chi) = \prod_{p \leq x} \left(1 - \frac{\chi(p)}{p}\right)^{-1} + O\left(\frac{1}{\log x}\right)$$

(cf. [1, §109]), whence

$$P_3(x) = \frac{\pi}{4} A_3 + O\left(\frac{1}{\log x}\right).$$

On the other hand, we have by a well-known theorem due to F. Mertens (cf. [1, §36])

$$P_1(x) P_3(x) = \frac{2e^{-C}}{\log x} + O\left(\frac{1}{\log^2 x}\right).$$

Received by the editors October 22, 1970.


Keywords and phrases. Dirichlet $L$-series. Mertens’ theorem.
The result (1) follows at once from (2) and (3).

We note that our asymptotic formula for \( P_1(x) \) will give a solution to a problem recently posed by D. Suryanarayana in [2].

References


Shinshu University, Matsumoto, Japan