FUNCTIONS WHICH ARE FOURIER-STIELTJES TRANSFORMS

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Abstract. Let $G$ be a locally compact abelian group, $\hat{G}$ the dual group, $M(G)$ the algebra of regular bounded Borel measures on $G$, and $M(G)^\wedge$ the algebra of Fourier-Stieltjes transforms. The purpose of this paper is to characterize those continuous functions on $\hat{G}$ which belongs to $M(X)^\wedge$, where $X$ is a closed subset of $G$ and $M(X) = \{\mu \in M(G) : \text{the support of } \mu \text{ is contained in } X\}$.

More precisely, we will prove the following theorem:

**Theorem.** Let $X$ be a closed subset of $G$ and $f$ a continuous function on $\hat{G}$. Then the following are equivalent:

(a) $f \in M(X)^\wedge$.

(b) $\{\lambda_n\} \subseteq M(\hat{G})$, $|\lambda_n(x)| \leq M$ for all $x \in \hat{X}$ and $\lambda_n(x) \to 0$ for all $x \in X$ implies $\int f d\lambda_n \to 0$.

The case where $f$ is assumed bounded and $X = G$ was proved by Ramirez in [2] by applying Grothendieck's completion theorem [1, p. 271] to the paired spaces $M(G)^\wedge$ and $M(\hat{G})$ under the pairing $(\mu, \lambda) = \int G f d\lambda$ where $\mu \in M(G)$ and $\lambda \in M(\hat{G})$.

We provide a short proof of the more general result using the well-known theorem of Eberlein (see, for example [3, p. 32]), which states that a continuous function $f$ on $\hat{G}$ is a Fourier-Stieltjes transform if and only if there exists a constant $A$ such that

$$\left| \sum_{i=1}^{n} c_i f(\gamma_i) \right| \leq A \| p \|_{\infty}, \quad \gamma_i \in \hat{G},$$

for every trigonometric polynomial $p$ on $G$ of the form

$$p(x) = \sum_{i=1}^{n} c_i \gamma_i(x), \quad x \in G.$$

**Proof.** Suppose $f = \hat{\mu}$ where $\mu \in M(X)$ and $\{\lambda_n\}$ satisfies the hypotheses of (b). Then by Fubini's theorem and the Lebesgue dominated convergence theorem, we have
Now assume that $f$ satisfies (b). We will first show that $f \in M(G)$. By Eberlein's theorem we must show that if $\{p_n\}$ is a sequence of trigonometric polynomials on $G$, say $p_n(x) = \sum c_{in} \gamma_{in}(x)$ where $x \in G$ and $\gamma_{in} \in \hat{G}$, with $p_n \to 0$ uniformly on $G$, then $\sum c_{in} f(\gamma_{in}) \to 0$ as $n \to \infty$.

Now let $\lambda_n = \sum c_{in} \delta_{in} \in M(\hat{G})$ where $\delta_{in}$ is the point mass at $\gamma_{in}$. Since $\lambda_n = p_n$ for every $n$, we have that $\{\lambda_n\}$ satisfies the hypotheses of (b) and hence

$$\sum c_{in} f(\gamma_{in}) = \int \, f d\lambda_n \to 0 \quad \text{as } n \to \infty.$$ 

Therefore $f = \bar{\mu}$ where $\mu \in M(G)$ and hence we need only show that the support of $\mu$ is contained in $X$. Suppose that this is not the case. Then the regularity of $\mu$ allows us to choose a compact set $E \subset X'$, the complement of $X$, such that $\mu(E) \neq 0$ and a sequence $\{U_n\}$ of open sets satisfying $E \subset U_n \subset X'$ and $\mu(U_n \setminus E) < n^{-1}$ for every $n$.

Now choose a sequence $\{\lambda_n\} \subset M(\hat{G})$ with $0 \leq \lambda_n \leq 1$, $\lambda_n = 0$ outside $U_n$, and $\lambda_n = 1$ on $E$ for every $n$. Clearly $\{\lambda_n\}$ satisfies the hypotheses of (b) and hence $\int \, f d\lambda_n \to 0$.

However,

$$\left| \int \, f d\lambda_n \right| = \left| \int_{U_n} \lambda_n d\mu \right| \geq \left| \int_E \lambda_n d\mu \right| - \left| \int_{U_n \setminus E} \lambda_n d\mu \right| \geq |\mu(E)| - n^{-1} \to |\mu(E)| \neq 0.$$

But this is a contradiction.

**Remark.** It should be noted that if the assumption of continuity is dropped and $X$ is replaced by $G$, then (b) will imply that $f \in M(\hat{G})$, where $\hat{G}$ is the Bohr compactification of $G$.

**References**


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