THE WEYL ESSENTIAL SPECTRUM

RICHARD BOULDIN

ABSTRACT. Using a modest geometric hypothesis the main theorem of these results classifies the Weyl essential spectrum and the Browder essential spectrum according to the standard terminology for the spectrum of a Hilbert space operator.

1. **Introduction.** The recent papers [1], [4], [6], [8] have classified the Weyl essential spectrum for various classes of operators on a Hilbert space. In [3] we classified the Wolf spectrum, the Weyl spectrum, and the Browder spectrum for a Hilbert space operator reduced by its finite dimensional geometric eigenspaces. We showed that our theorems contained the results in the above-mentioned papers and finally we listed a number of applications. Our purpose in the present note is to examine a condition which is alternative to the hypothesis that each finite dimensional geometric eigenspace reduces the operator. This alternative condition is properly weaker, although somewhat more complicated. We deduce a simplified form of the main result of [3] using this alternative hypothesis; neither do we repeat the applications of the main result nor do we relate the main theorem to the above-mentioned papers since that is the same as in [3]. We study the alternative hypothesis in order to show that it is properly weaker than the previous hypothesis and to show that it is easily verified.

2. **Preliminaries.** Throughout this note we shall use "operator" to mean a closed linear operator defined on a vector space which is dense in the fixed underlying Hilbert space H. If T is such an operator and H_0 is a subspace of H invariant under T then T/H_0 denotes the restriction of T to H_0. The conjugate of the complex number z is written z^* and we write the scalar operator zI as simply z. We say λ is an isolated point of $\sigma(T)$ to mean there is no sequence $\{\lambda_n : \lambda_n \neq \lambda, \lambda_n \in \sigma(T)\}$ which converges to λ and we denote closure of the set S by S^-.

An operator T is said to be Fredholm if dimension [kernel T] $< \infty$, TH is closed, and codimension $TH < \infty$. If T is Fredholm then the

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index of T is $[\dim \ker T - \operatorname{codim} TH]$. We recall the following defini-
tions of essential spectrum which have been given for closed operators
in a Banach space. A point λ from the spectrum of T, i.e. $\lambda \in \sigma(T)$, is
in the essential spectrum of Wolf provided that either $(T-\lambda)H$ is
not closed or else it has infinite codimension. The essential spectrum
of Weyl is $\{\lambda \in \sigma(T) : (T-\lambda)$ is not a Fredholm operator with index
equal to $0\}$; the essential spectrum of Browder is $\{\lambda \in \sigma(T) :$ either
$(T-\lambda)$ is not a Fredholm operator with index equal to 0 or λ is not
an isolated point of $\sigma(T)\}$. If these three sets are ordered by inclusion
then the above enumeration is clearly nondecreasing.

In order to save space we are going to give a name to the condition
which is our basic hypothesis. The condition is the following: corre-
sponding to $\lambda \in \sigma(T)$ there is a number less than 1, say δ, such that
$|\langle f, g \rangle| \leq \delta$ whenever $f \in \ker (T-\lambda)$, $\|f\| = 1 = \|g\|$, and g is an eigen-
vector for some eigenvalue distinct from λ. When the condition is
satisfied we shall say "$\ker (T-\lambda)$ is not an asymptotic eigenspace."
Of course if the condition fails to hold then "$\ker (T-\lambda)$ is an asymp-
totic eigenspace."

3. **Classification of essential spectrum.** We prove a basic fact about
the spectrum of a Hilbert space operator.

THEOREM 1. *Let T be an operator on H such that* $\ker (T-\lambda)$ *is not
an asymptotic eigenspace of T and let* $\{\lambda_n\}$ *be a sequence such that
$\lambda_n \neq \lambda$, $\lambda_n \in \sigma(T)$, and $\lambda_n \to \lambda$. If for each $n = 1, 2, \cdots$ there exists a
unit vector f_n such that $f_n \in \ker(T-\lambda_n)$ then $(T-\lambda)H$ is not closed.*

PROOF. Take a sequence of unit vectors such that $\|(T-\lambda_n)f_n\|$
$= 0$ and decompose f_n into $f_n' + f_n''$ where $f_n' \in \ker (T-\lambda)$ and
$f_n'' \in [\ker (T-\lambda)]^{\perp}$. If P is the orthogonal projection onto $\ker (T-\lambda)$
then $f_n' = Pf_n$ and $f_n'' = (I-P)f_n$. Because $\ker (T-\lambda)$ is not an asymp-
totic eigenspace of T for each n and any unit vector $g \in PH$ we have
$|\langle f_n, g \rangle| \leq \delta$; in particular this is true for $g = f_n'$, that is

$$\delta \geq |\langle f_n, f_n' \rangle| = |\langle f_n, Pf_n \rangle| = |\langle f_n, P^2 f_n \rangle|$$

$$= |\langle Pf_n, Pf_n \rangle| = \|f_n'\|^2.$$

Thus $\|f_n''\|^2 = 1 - \|f_n'\|^2 \geq 1 - \delta > 0$. Now we note the following sim-
plifications

$$(T - \lambda)f_n'' = (T - \lambda)(f_n - f_n') = (T - \lambda)f_n,$$

$$\|(T - \lambda)f_n''\| \leq \|(T - \lambda_n)f_n\| + \|(\lambda_n - \lambda)f_n\|$$

$$= \|(T - \lambda_n)f_n\| + |\lambda_n - \lambda|.$$

Consequently $\|(T-\lambda)f_n''\|\to 0$ as $n\to\infty$. We may replace the original sequence $\{f_n\}$ with the sequence $\{f_n''/\|f_n''\|\}$ and thus we may assume that $\|(T-\lambda)f_n\|\to 0$ and $f_n\in[\ker(T-\lambda)]^\perp$.

It follows that $(T-\lambda)^+(T-\lambda)f_n=f_n$ where $(T-\lambda)^+$ is the linear transformation inverse to $(T-\lambda)$: $[\ker(T-\lambda)]^\perp\to[(T-\lambda)H]^-$. If $(T-\lambda)H$ is closed then $(T-\lambda)^+$ is bounded by application of the closed graph theorem, noting that $(T-\lambda)/[\ker(T-\lambda)]^\perp$ is a closed operator. Consequently if $(T-\lambda)H$ is closed then $(T-\lambda)^+$ is bounded and $\|(T-\lambda)^+(T-\lambda)f_n\|\to 0$ since $\|(T-\lambda)f_n\|\to 0$. However this is nonsense since $\|(T-\lambda)^+(T-\lambda)f_n\|=\|f_n\|=1$ and hence $(T-\lambda)H$ is not closed.

The following lemma is a special case of a well-known theorem.

LEMMA 1. *Let $\{\lambda_n\}$ be a sequence such that $\lambda_n\in\sigma(T)$, $\lambda_n\neq\lambda$, and $\lambda_n\to\lambda$. If for each n the operator $(T-\lambda_n)$ is not a Fredholm operator then $(T-\lambda)$ is not a Fredholm operator.*

PROOF. Because $(T-\lambda_n)$ converges to $(T-\lambda)$ in the operator norm we would contradict Theorem 5.17, p. 235, of [7] if $(T-\lambda)$ were a Fredholm operator.

We now conclude our consideration of nonisolated points of the spectrum.

THEOREM 2. *Let T be an operator on H such that $\ker(T-\lambda)$ is not an asymptotic eigenspace. If there exists a sequence, say $\{\lambda_n\}$, such that $\lambda_n\in\sigma(T)$, $\lambda_n\neq\lambda$, and $\lambda_n\to\lambda$ then $(T-\lambda)$ is not a Fredholm operator.*

PROOF. For $\lambda_n\in\sigma(T)$ there are only three possibilities: (a) $(T-\lambda_n)$ is not one-to-one, (b) $(T-\lambda_n)$ is one-to-one but $(T-\lambda_n)H$ is not closed, (c) $(T-\lambda_n)$ is one-to-one with closed range but $[(T-\lambda_n)H]^\perp\neq\{0\}$. Otherwise we could apply the closed graph theorem to see that $(T-\lambda_n)^{-1}$ is everywhere defined and bounded.

Now let $\{\lambda_n\}$ be any sequence such that $\lambda_n\in\sigma(T)$, $\lambda_n\neq\lambda$, and $\lambda_n\to\lambda$. Either there is an infinite subset of $\{\lambda_n\}$ satisfying (a), in which case we apply Theorem 1 to see that $(T-\lambda)H$ is not closed, or else there is an infinite subset of $\{\lambda_n\}$ such that for each μ in that set $(T-\mu)$ satisfies either (b) or (c). If either (b) or (c) is satisfied by $(T-\mu)$ then this operator is not Fredholm and we may apply Lemma 1 to conclude that $(T-\lambda)$ is not Fredholm.

We are able to complete our consideration of isolated points of the spectrum with the following lemma. For a definition of algebraic multiplicity see pp. 178–181 of [7].

LEMMA 2. *Let T be an operator on H with λ an isolated point of $\sigma(T)$.*

$(T-\lambda)H$ *is closed with finite codimension if and only if λ is an isolated eigenvalue with finite algebraic multiplicity.*

PROOF. See Lemma 3 and the second and third paragraphs of the proof of Theorem 3 in [3].

Now we can state the main result of this note.

THEOREM 3. *If each finite dimensional subspace* ker $(T-z)$, *for any complex number z, is not an asymptotic eigenspace then the Weyl spectrum of T coincides with the Browder spectrum of T which coincides with the points of $\sigma(T)$ that are not isolated eigenvalues with finite algebraic multiplicity.*

PROOF. Apply Lemma 2 to any isolated point of $\sigma(T)$ and apply Theorem 2 to any nonisolated λ such that ker $(T-\lambda)$ is finite dimensional. If ker $(T-\lambda)$ is infinite dimensional then obviously $(T-\lambda)$ is not Fredholm.

4. An asymptotic eigenspace. In this section we shall study the condition "ker $(T-\lambda)$ is not an asymptotic eigenspace." We shall show that our condition follows from several mild hypotheses. We recall that the angle between two subspaces H_1 and H_2 is the number $\theta \in [0, \pi]$ such that $\cos \theta = \sup\langle f, g\rangle$ where the supremum is taken over all unit vectors f and g such that $f \in H_1$ and $g \in H_2$. We say that two subspaces, H_1 and H_2, are complementary provided that $H = H_1 \oplus H_2$.

THEOREM 4. *If* ker $(T-\lambda)$ *has an invariant complementary subspace say H_1 with a positive angle between H_1 and* ker $(T-\lambda)$ *then* ker $(T-\lambda)$ *is not an asymptotic eigenspace.*

PROOF. Note that $\sigma(T) = \sigma(T/\text{ker }(T-z)) \cup \sigma(T/H_1)$ and $\sigma(T/\text{ker }(T-z)) = \{z\}$. Let μ be an eigenvalue of T distinct from z and choose $g \in$ ker $(T-\mu)$. Write g as $g'+g''$ where $g' \in$ ker $(T-z)$ and $g'' \in H_1$ and note that $(T-\mu)g' \in$ ker $(T-z)$ and $(T-\mu)g'' \in H_1$. Since $(T-\mu)g=0$ it must be that $(T-\mu)g' = -(T-\mu)g''$ is in the intersection of ker $(T-z)$ and H_1; hence $(T-\mu)g' = 0 = (T-\mu)g''$. If $g' \neq 0$ then $\mu \in \sigma(T/\text{ker }(T-z))$ which is a contradiction. Thus $g = g'' \in H_1$ and we conclude that ker $(T-\mu) \subset H_1$. Since H_1 has a positive angle to ker $(T-z)$ we conclude that ker $(T-\lambda)$ is not an asymptotic eigenspace.

COROLLARY 1. *If* ker $(T-\lambda)$ *reduces T then it is not an asymptotic eigenspace.*

The next theorem will justify the terminology "asymptotic eigen-space."

THEOREM 5. *If* ker $(T-\lambda)$ *is a finite dimensional asymptotic eigenspace then there exists a sequence of eigenvalues* $\{\lambda_n\}$ *with* $\lambda_n \neq \lambda$ *and a sequence of unit vectors* $\{g_n\}$ *with* $g_n \in$ ker $(T-\lambda_n)$ *such that* $\{g_n\}$ *converges in norm to a unit vector* f *where* $f \in$ ker $(T-\lambda)$. *Furthermore, if* T *is bounded then* $\lambda_n \to \lambda$.

PROOF. Because $(T-\lambda)$ is an asymptotic eigenspace for each n we can find an eigenvalue λ_n and unit vectors $g_n \in$ ker $(T-\lambda_n)$, $f_n \in$ ker $(T-\lambda)$ such that $\langle g_n, f_n \rangle \geq 1 - 1/n$. Since $\langle g_n, f_n \rangle \leq \|g_n\| \|f_n\|$ we see that $\langle g_n, f_n \rangle \to 1$; using the fact that $\|g_n\|^2 = 1 = \|f_n\|^2$ we conclude that $\langle g_n, g_n \rangle - \langle g_n, f_n \rangle = \langle g_n, g_n - f_n \rangle \to 0$ and $\langle f_n, g_n \rangle - \langle f_n, f_n \rangle = \langle f_n, (g_n - f_n) \rangle \to 0$. Thus $\|g_n - f_n\|^2$ converges to 0. Because the unit ball in ker $(T-\lambda)$ is compact $\{f_n\}$ has a subsequence converging to some $f \in$ ker $(T-\lambda)$. Clearly $\{g_n\}$ converges to f and if T is bounded then Tg_n converges to Tf and $\lambda_n = \langle Tg_n, g_n \rangle$ converges to $\lambda = \langle Tf, f \rangle$.

This gives rise to another condition which is sufficient for ker $(T-\lambda)$ not to be a finite dimensional asymptotic eigenspace.

COROLLARY 2. *Let* T *be a bounded operator. If* λ *is not a limit point for the set of eigenvalues of* T *then* ker $(T-\lambda)$ *is not an asymptotic eigenspace.*

The final two corollaries were called to the author's attention by the referee.

COROLLARY 3. *If* T *has the property that* ker $(T-\lambda)$ *is orthogonal to* ker $(T-\mu)$ *whenever* $\lambda \neq \mu$ *then* T *has no asymptotic eigenspaces.*

COROLLARY 4. *If* T *has the property that the numerical radius of any restriction of* T *to an invariant subspace is equal to the spectral radius of that restriction then* T *has no asymptotic eigenspaces.*

PROOF. Use Lemma 5, p. 191, of [2] along with Corollary 3 above.

REMARK. We note that Theorem 1 is not necessarily true when ker $(T-\lambda)$ is an asymptotic eigenspace. Let S^* be the adjoint of the unilateral shift and recall that each λ such that $|\lambda| < 1$ is a simple eigenvalue for S^* although S^*H is clearly equal to H and thus S^*H is closed.

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UNIVERSITY OF GEORGIA, ATHENS, GEORGIA 30601