KLEIN BOTTLES IN CIRCLE BUNDLES

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ABSTRACT. We prove that the Klein bottle embeds in the total space $E$ of an orientable $S^1$-bundle over an orientable 2-manifold $M$ if and only if $M = S^2$ and $E = S^1 \times S^2$ or the lens space $L(4, 1)$.

In this note we apply results of [1] to generalize a result given there concerning the embedding of the Klein bottle.

PROPOSITION. The Klein bottle embeds in the total space $E$ of an orientable $S^1$-bundle over an orientable 2-manifold $M$ if and only if $M = S^2$ and $E = S^1 \times S^2$ or the lens space $L(4, 1)$.

To show $M = S^2$ we use the following result of [1]:

THEOREM [1, §4.1]. Let $i: K \to E$ be an embedding of a nonorientable $(n-1)$-manifold $K$ in an orientable $n$-manifold $E$. Suppose that $a \in \pi_1(K)$ reverses orientation. Then for $\beta \in \pi_1(E)$, $\beta^{-1} i_*(a) \beta \in i_*(\pi_1(K))$ implies $\beta \in i_*(\pi_1(K))$.

Assume $M \neq S^2$, so $\pi_2(M) = 0$. In the exact sequence of the fibration

$$\cdots \to 0 \to \pi_1(S^1) \to \pi_1(E) \to \pi_1(M) \to 0$$

the generator of $\pi_1(S^1)$ is mapped to an element $g$ in the center of $\pi_1(E)$. (Since $E$ is trivial over the 1-skeleton of $M$, the inverse image of any circle in $M$ is a torus in $E$. Hence $g$ commutes with a basis for $\pi_1(E)$.) By the theorem $g$ is in the image of $i_*$. Let $\pi_1(K) = \{\alpha, \beta: \alpha \beta \alpha^{-1} = \beta^{-1}\}$; $\alpha$ is the orientation reversing element. Then $g = i_*(\alpha^j \beta^k)$ for some integers $j, k$. Since $\alpha \beta \alpha^{-1} = \beta^{-k}$, we have

$$g i_*(\alpha^{-j} \beta^{k}) g i_*(\alpha^{-j} \beta^{k}) = i_*(\alpha^j \beta^k \alpha^{-j} \beta^k \alpha^{-j} \beta^k) = 1.$$ 

Therefore $g^2 = i_*(\alpha^{2j})$. $p_1(g) = 0$ and $\pi_1(M)$ is torsion free, so $p_1i_*(\alpha) = 0$. Therefore $i_*(\alpha) = g^n$ and is in the center of $\pi_1(E)$. But then by the theorem $i_*$ is onto. Thus $p_1i_*(\beta)$ generates $\pi_1(M)$ which contradicts $M \neq S^2$.

To complete the proof of the proposition recall that the total space $E$ of an orientable $S^1$-bundle over $S^2$ is the lens space $L(k, 1)$ or $S^1 \times S^2$ (the case $k = 0$). By [1, §6] a nonorientable surface of genus $g$ embeds

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in $L(k, 1)$ if and only if $k$ is even, $g \equiv k/2 \pmod{2}$, and $g \geq k/2$. Thus the Klein bottle, which has genus 2, embeds only in $L(4, 1)$ and $S^1 \times S^2$.

If $S^1 \times S^2$ is pictured as a family of 2-spheres parameterized by $\theta$, $0 \leq \theta < 2\pi$, then the surface swept out by a meridian rotated about the poles by $\theta/2$ is a Klein bottle.

In the $x, y$-plane let $S$ be the square with vertices at $(\pm 1, 0)$ and $(0, \pm 1)$. $L(4, 1)$ is obtained from the suspension from $(0, 0, 1)$ of $S$ in $\mathbb{R}^3$ by identifying certain points of the boundary, cf. [2, p. 223]. The surface $z = xy$ gives an embedding of the Klein bottle.

REFERENCES


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