PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, Number 2, May 1971

ON ORDERED POLYCYCLIC GROUPS

R. J. HURSEY, JR.

ABSTRACT. It has been asserted that any (full) order on a torsion-free, finitely generated, nilpotent group is defined by some F-basis of G and that the group of o-automorphisms of such a group is itself a group of the same kind: Examples provided herein demonstrate that both of these assertions are false; however, it is proved that the group of o-automorphisms of an ordered, polycyclic group is nilpotent by abelian, and polycyclic.

1. **Introduction.** It is well known that if G is a torsion-free, finitely generated, nilpotent group, then G possesses a central series $\{1\} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_m = G$ such that F_{i+1}/F_i is an infinite cyclic group, $i=0, 1, \cdots, m-1$. Ree (see [5] and [6]) calls such a series an "F-series of G." It is also clear that if $F_{i+1}/F_i = \langle x_{i+1}F_i \rangle$, where $x_{i+1} \in F_{i+1}$ for $i=0, 1, \cdots, m-1$, then each element g of G can be written uniquely in the form $g = x_1^{e_1}x_2^{e_2} \cdots x_m^{e_m}$, where $e_1, e_2, \cdots, e_m$ are integers. Ree [6] calls the elements $x_1, x_2, \cdots, x_m$ an "F-basis of G." It follows easily that the torsion-free, finitely generated, nilpotent group G can be ordered lexicographically with respect to $e_1, \cdots, e_m$ as follows: For $g_1 = x_1^{e_1}x_2^{e_2} \cdots x_m^{e_m}$, $g_2 = x_1^{f_1}x_2^{f_2} \cdots x_m^{f_m}$, we put $g_1 \leqq g_2$ if and only if $e_i = f_i$ for $i=1, 2, \cdots, m$, or $e_t < f_t$ for some t such that $1 \leqq t \leqq m$ and $e_i = f_i$ for $i=1, 2, \cdots, t-1$. This lexicographic order on G is said, by Ree [5], to be "defined" by the F-basis $x_1, x_2, \cdots, x_m$.

These concepts are used by Ree in [6], where it is asserted that any (full) order on a torsion-free, finitely generated, nilpotent group is defined by some F-basis of G, and in the proof of Theorem 2 [6], which asserts that the group of o-automorphisms of a torsion-free, finitely generated, nilpotent group is itself a torsion-free, finitely generated, nilpotent group. Both of these assertions, as shown by Examples 1 and 2 of this note, are false. While it is not possible to establish a result as strong as the one suggested by Theorem 2 of [6], we can prove

THEOREM. *If G is an ordered, polycyclic group and if Δ denotes the group of o-automorphisms of G, then Δ is nilpotent by abelian, and, moreover, Δ is polycyclic.*

Received by the editors September 15, 1970.

AMS 1969 *subject classifications*. Primary 0675, 2022; Secondary 2018, 2040.

Key words and phrases. Polycyclic group, nilpotent group, cyclic normal series, F-series, F-basis, length of a polycyclic group.

2. **Definitions and notations.** If G is a group on which there can be defined a (full) order relation $\leqq$ with the property that $a, b, x, y \in G$ and $a \leqq b$ imply $xay \leqq xby$, then G is said to be an *ordered group* and $\leqq$ is said to be an *order* on G. Associated with an order $\leqq$ on G is the *positive cone* $P(G)$ of G, $P(G) = \{x \mid x \in G \text{ and } 1 \leqq x\}$. It follows that the subset $P(G)$ of the ordered group G has the following properties:
(i) $P(G) \cap P^{-1}(G) = \{1\}$;
(ii) $P(G)P(G) \subseteq P(G)$;
(iii) $x^{-1}P(G)x \subseteq P(G)$; and
(iv) $P(G) \cup P^{-1}(G) = G$.
Conversely, if G is a group which possesses a subset $P(G)$ with properties (i)–(iv), then G is an ordered group with respect to the relation $\leqq$ given by

$$a \leqq b \text{ if and only if } a^{-1}b \in P(G).$$

A subgroup C of a group G ordered with respect to $\leqq$ is *convex* (with respect to $\leqq$) if $g \in G$, $c \in C$, and $1 \leqq g \leqq c$ imply $g \in C$.

If $D \subset C$ are convex subgroups of the ordered group G with the property that no convex subgroup of G lies strictly between D and C, then $D \prec C$ is a *jump* in the chain of convex subgroups of G.

If G and H are ordered groups and f is a mapping of G into H, then f is an *o-homomorphism* of G into H if f is a group homomorphism of G into H and f is order-preserving in the sense that $a, b \in G$ and $a \leqq_1 b$ imply $f(a) \leqq_2 f(b)$, where $\leqq_1$ and $\leqq_2$ denote the orders on G and H, respectively. Furthermore, if f is a one-to-one *o-homomorphism* of G onto H and if f^{-1} is an o-homomorphism of H onto G, then f is an *o-isomorphism* of G onto H. An o-isomorphism of an ordered group G onto G is an *o-automorphism* of G.

If G is a group, then the series $\{1\} = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = G$ is a *cyclic normal* (*invariant*) *series* of G if A_i is a normal subgroup of $A_{i+1}(G)$ and A_{i+1}/A_i is cyclic, $i = 0, 1, \cdots, n-1$. A group G is *polycyclic* if G possesses a cyclic normal series. Finally, by the *length* of a polycyclic group G, we mean the number of infinite cyclic factors A_{i+1}/A_i in any cyclic normal series of G. It is well known that the length of a polycyclic group is an invariant for that group.

3. **Proofs.**

EXAMPLE 1. Let G be the subgroup of the additive group of reals which is generated by $\{1, \sqrt{2}\}$; i.e., $G = \langle 1 \rangle + \langle \sqrt{2} \rangle$. By restricting the natural Archimedean order on the reals to G, G is an Archimedean ordered, finitely generated, abelian group, whence G possesses no proper, nontrivial, convex subgroups.

Let $\{1\} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_m = G$ be an F-series of G with corresponding F-basis $f_1, f_2, \cdots, f_m$. Then, as G is polycyclic of length two, $m=2$, so that $\{1\}=F_0 \subset F_1 \subset F_2 = G$, $F_1 = \langle f_1 \rangle$ and $F_2/F_1 = \langle f_2 F_1 \rangle$. Thus, each element $g \in G$ can be written uniquely in the form $e_1 f_1 + e_2 f_2 = g$, where e_1, e_2 are integers. Let $\leqq_1$ denote the order on G defined by the F-basis f_1, f_2.

Now suppose that $g = e_1 f_1 + e_2 f_2$ and $0 \leqq_1 e_1 f_1 + e_2 f_2 \leqq e_2' f_2$, where e_1, e_2, and e_2' are integers. Then $0 \leqq_1 -e_1 f_1 + (e_2' - e_2) f_2$. Thus, as $0 \leqq e_1$ and $0 \leqq -e_1$, $e_1 = 0$ and, hence, $g = e_2 f_2 \in \langle f_2 \rangle$. Therefore, $\langle f_2 \rangle$ is a proper, nontrivial, convex subgroup of G with respect to $\leqq_1$, whence the F-basis f_1, f_2 cannot define the given Archimedean order on G.

EXAMPLE 2. Let $G = \langle a_1 \rangle + \langle a_2 \rangle + \langle a_3 \rangle$, where $\langle a_1 \rangle$ is an infinite cyclic group and where $\langle a_2 \rangle + \langle a_3 \rangle$ is isomorphic to the subgroup $H = \langle 1 \rangle + \langle (1/2)(1+\sqrt{5}) \rangle$ of the additive group of reals. Let $P(G) = \{ na_1 \mid n \text{ is a nonnegative integer} \} \cup \{ x \mid x \in G - \langle a_1 \rangle, x = ra_1 + sa_2 + ta_3, \text{ and } s + (t/2)(1+\sqrt{5}) > 0 \}$. It readily follows that $P(G)$ defines an order on G, say $\leqq$, where for $x, y \in G$, $x \leqq y$ if and only if $(-x+y) \in P(G)$. It is easy to see that $0 < a_1$, $0 < a_2$, and $0 < a_3$; also, $P(G)$ is not an Archimedean order on G as $a_1 \ll a_2$ and $a_1 \ll a_3$ (i.e., $na_1 < a_2$, $na_1 < a_3$ for all integers n).

We now define two o-automorphisms of G:

(i) $d: a_1 \to a_1,\ a_2 \to a_3,\ a_3 \to a_2 + a_3$,

(ii) $v: a_1 \to a_1,\ a_2 \to a_2 + ma_1,\ a_3 \to a_3 + na_1$,

where m, n are arbitrary nonzero integers.

It follows easily that d and v are automorphisms of G. We now show that d is order-preserving: Let $x = ra_1 + sa_2 + ta_3$ and suppose $0 < x$. If $s = t = 0$, then $x^d = ra_1 > 0$. Suppose, therefore, that $0 \neq s$ or $0 \neq t$. Then, without loss of generality, we may assume $r = 0$, so that $x = sa_2 + ta_3$. Then $0 < x$ is equivalent to $0 < s + (t/2)(1+\sqrt{5})$, which is equivalent to $(-2s)/(1+\sqrt{5}) < t$. On the other hand, $0 < (sa_2 + ta_3)^d = ta_2 + (s+t)a_3$ is equivalent to $3t + t\sqrt{5} > -s(1+\sqrt{5})$, which is true if and only if $t > -s(1+\sqrt{5})/(3+\sqrt{5}) = -2s/(1+\sqrt{5})$. Thus, d is order-preserving. An analogous argument establishes that v is also order-preserving, whence d and v are o-automorphisms of G.

Let $\Delta = \langle d, v \rangle$, so that Δ is a subgroup of the group of o-automorphisms of G. It follows easily that $v = [v, d]^d$, whence $v \neq 1$ belongs to each term of the lower central series of Δ. Therefore, Δ is not nilpotent, and, as Δ is a subgroup of the group of o-automorphisms of G, the group of o-automorphisms of the torsion-free, finitely generated, abelian group G is not nilpotent.

We proceed now with the

PROOF OF THE THEOREM. Let $\{1\} \prec C_1 \prec C_2 \cdots \prec C_n = G$ be the chain of convex subgroups of G with respect to the given order on G.

We observe here that as G satisfies the maximal condition for subgroups, this chain is necessarily of finite length and that this chain is an invariant series of G (see [3]). Let $\theta \in \Delta$. Then θ induces an o-automorphism on the ordered group C_i/C_{i-1}, for $i=1, 2, \cdots, n$, given by $(cC_{i-1})^{\theta'} = c^{\theta}C_{i-1}$.

For each i such that $1 \leqq i \leqq n$, let Δ_i denote the group of all o-automorphisms $\theta \in \Delta$ such that θ' centralizes C_i/C_{i-1}, i.e., such that $(cC_{i-1})^{\theta'} = c^{\theta}C_{i-1} = cC_{i-1}$. Then Δ_i is a normal subgroup of Δ for each i, and Δ/Δ_i is isomorphic to a subgroup of the group of o-automorphisms of C_i/C_{i-1}. But, each C_i/C_{i-1} is an Archimedean ordered group (see [1, p. 50]), whence the group of o-automorphisms of C_i/C_{i-1} is isomorphic to a subgroup of the multiplicative group of positive real numbers (see [1, Corollary 3, p. 47]). Thus, Δ/Δ_i is abelian for $i = 1, 2, \cdots, n$. Hence, Δ/Δ_0 is abelian, where $\Delta_0 = \bigcap_{i=1}^{n} \Delta_i$. Note that Δ_0 centralizes C_i/C_{i-1} for each i, so that $[C_i, \Delta_0] \subseteq C_{i-1}$, $i = 1, 2, \cdots, n$. Thus, by a result of P. Hall (see [2, Corollary to Theorem 3.8, p. 10]), Δ_0 is nilpotent. Therefore, Δ is nilpotent by abelian.

Smirnov [7] has proved that, for a polycyclic group H, every abelian subgroup of $\mathrm{Aut}(H)$ is finitely generated, whereas Mal'cev [4] has proved that any solvable group, all of whose abelian subgroups are finitely generated, is polycyclic. These results prove that Δ is polycyclic.

As a final remark, I should like to thank Professors R. D. Bercov and A. H. Rhemtulla for their invaluable assistance in the preparation of this paper and for the many fruitful discussions while this author was a student at the University of Alberta.

REFERENCES

1. L. Fuchs, *Partially ordered algebraic systems*, Pergamon Press, New York; Addison-Wesley, Reading, Mass., 1963. MR **30** #2090.

2. P. Hall, *Nilpotent groups*, Lecture Notes, Fourth Canadian Math. Congress, 1957.

3. R. J. Hursey, Jr., and A. H. Rhemtulla, *Ordered groups satisfying the maximal condition locally*, Canad. J. Math. **22** (1970), 753–758.

4. A. I. Mal'cev, *On certain classes of infinite solvable groups*, Mat. Sb. **28** (**70**) (1951), 567–588; English transl., Amer. Math. Soc. Transl. (2) **2** (1956), 1–21. MR **13**, 203.

5. R. Ree, *On ordered, finitely generated, solvable groups*, Trans. Roy. Soc. Canad. Sect. III (3) **48** (1954), 39–42. MR **16**, 792.

6. ———, *The existence of outer automorphisms of some groups*. II, Proc. Amer. Math. Soc. **9** (1958), 105–109. MR **20** #912.

7. D. M. Smirnov, *On a class of infinite solvable groups*, Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki **5** (1954), 57–60. (Russian) MR **17**, 709.

EAST CAROLINA UNIVERSITY, GREENVILLE, NORTH CAROLINA 27834