NONLINEAR AND DIRECTION CONNECTIONS

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Abstract. Nonlinear connections and direction connections are two types of connections arising in Finsler geometry. In his work on generalized sprays, P. Dazord showed that there is a relationship between these two types (nonlinear connections were called sections by him). This relationship has also been used by J. Grifone in a work on prolongation of direction connections. In this paper we examine this relationship in a general setting. In particular, we show that E. Cartan's condition "D" is necessary and sufficient for a direction connection to define a nonlinear one. Also, we prove a nonuniqueness result for direction connections associated to a given nonlinear one.

1. Connections on vector-bundles. We first recall our definition of a connection on a vector bundle [5], [6]. For a smooth \((C^\infty)\) vector bundle \(p:E\rightarrow M\), set \(E_0=E|_0\) and \(p_0=p|_E\). The bundle \(p^{-1}E\) over \(E\) is canonically isomorphic to the bundle \(VE\) of vertical vectors in the tangent bundle \(TE\) of \(E\). Hence we have the exact sequence

\[
\begin{array}{c}
0 \rightarrow p^{-1}E \rightarrow TE \rightarrow p^{-1}TM \rightarrow 0 \\
\end{array}
\]

of vector bundles over \(E\), where \(J\) corresponds to the inclusion map \(VECTE\) and \(p'\) is essentially the tangent map \(p_*:TE\rightarrow TM\).

A smooth nonlinear connection on the vector bundle \(p:E\rightarrow M\) is a smooth splitting of (1) over \(E_0\). Since \(TE|_E=E_0=TE_0\) and \(p^{-1}E|_E=p_0^{-1}E\), such a splitting is given by a smooth linear map \(V:TE_0\rightarrow p_0^{-1}E\) (i.e. continuous linear on the fibres), satisfying the equation \(VJ=\text{id}\). The splitting can be conveniently described by its connection map \(D:TE_0\rightarrow E\) defined as \(D=r\circ V\), where \(r:p^{-1}E\rightarrow E\) is the canonical surjection over \(p\). \(D\) is continuous linear on the fibres and is smooth.

The connection on \(p:E\rightarrow M\) is homogeneous (resp. linear) if the map \(D\) is homogeneous of degree 1 (resp. linear) on the \(p_*\) fibres of \(TE\). For a linear connection, the splitting of (1) automatically extends to all of \(E\); in fact, a linear connection can be defined as a splitting of (1) which is smooth over all of \(E\).

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Let us now introduce local coordinates. Suppose locally $TM$ and $E$ are isomorphic to product bundles $U \times B$ and $U \times E$, respectively, where $B$ and $E$ are real Banach spaces (the reader should substitute $B = \mathbb{R}^n$, $E = \mathbb{R}^m$ if he is not familiar with calculus in Banach spaces) and $U \subseteq B$ is an open set. Then $TE_0$ is locally isomorphic to the product bundle $(U \times E_0) \times B \times E$ (where $E_0 = E - 0$), and fiberwise for each $(x, a) \in U \times E$, $D$ is defined by a map $D_{(x,a)} : B \times E \rightarrow E$ of the form

\begin{equation}
D_{(x,a)}(\lambda, b) = b + \omega(x, a) \lambda.
\end{equation}

Here $\omega : U \times E_0 \rightarrow L(B, E)$ is a smooth map ($L$ denotes a space of continuous linear maps). $\omega$ is called a local component of the connection.

The connection is homogeneous (resp. linear) iff each $\omega$ is homogeneous of degree 1 (resp. continuous linear) in the second variable, $a$. (Hence for a linear connection we can define $\omega(x, 0) = 0$ and obtain a smooth map on $U \times E$.)

2. Linear connections on $p_0^{-1}E \rightarrow E_0$. Next we consider a linear connection on the vector bundle $p_0^{-1}E \rightarrow E_0$ and define its torsion.

Since $p_0^{-1}E$ is locally isomorphic to $(U \times E_0) \times E$, and $T(p_0^{-1}E)$ to $((U \times E_0) \times E) \times (B \times E) \times E$, the connection map $\mathcal{D} : T(p_0^{-1}E) \rightarrow p_0^{-1}E$ of such a connection is, fiberwise for each $((x, a), b) \in (U \times E_0) \times E$, given by a map $\mathcal{D}_{((x,a),b)} : (B \times E) \times E \rightarrow E$ of the form

\begin{equation}
\mathcal{D}_{((x,a),b)}((\lambda, c), d) = d + \Omega((x, a), b)(\lambda, c).
\end{equation}

$\Omega$ is continuous linear in $(\lambda, c)$ and, the connection being linear, also in $b$. Hence $\mathcal{D}$ must actually be locally of the form

\begin{equation}
\mathcal{D}_{((x,a),b)}((\lambda, c), d) = d + \Gamma((x,a),b)(\lambda) + C_{(x,a)}(b, c),
\end{equation}

where $\Gamma$, $C$ are smooth maps of $U \times E_0$ into $L^2(E, B; E)$ and $L^2(E, E; E)$, respectively ($L^2$ denotes a space of continuous bilinear maps).

Although the two parts $\Gamma$ and $C$ of the local component of $\mathcal{D}$ do not, of course, transform together as a tensor under a change of coordinates, it happens that the $C$ part does define a tensor, which corresponds to the Finsler torsion tensor of [1]. (Since $p^{-1}E \approx VE$, one can transcribe the coordinate change equation of [6, p. 1127] to the present notation and set $\mu = 0$ to see this fact.) Thus there is a smooth section $\mathfrak{T}$ of the bundle of bilinear maps from $p_0^{-1}E$ to $p_0^{-1}E$, which is locally given by maps $\mathfrak{T}_{(x,a)} \subseteq L^2(E, E; E)$ with $\mathfrak{T}_{(x,a)}(b, c) = C_{(x,a)}(b, c)$. We call $\mathfrak{T}$ the torsion tensor of the given linear connection on $p_0^{-1}E \rightarrow E_0$.

3. Direction connections. We now define direction connections for vector bundles. Namely, a direction connection for the vector bundle...
\(\mathcal{D} \circ (f)_\ast = f \circ \mathcal{D}\)

for all such diffeomorphisms \(f\).

Next we derive the local coordinate description of direction connections. Namely, consider a local representation \(f(x, a, b) = (x, \phi(x)a, b)\) of a diffeomorphism \(f\) defined above, where \(\phi : U \rightarrow \mathbb{R}\) is a smooth nonvanishing function. An easy calculation shows (4) to mean

\[
\Gamma_{(x,a)}(b, \lambda) + C_{(x,a)}(b, c) = \Gamma_{(x,\phi(x)a)}(b, \lambda) + C_{(x,\phi(x)a)}(b, \phi(x)c) + C_{(x,\phi(x)a)}(b, \phi'(x)(\lambda)a).
\]

Putting \(c = 0\) and then setting \(\phi\) constant, we see that (5) is equivalent to the two conditions that

\[
\Gamma_{(x,a)}(b, \lambda) \text{ is homogeneous of degree 0 in } a, \text{ and}\\
C_{(x,a)}(b, c) = C_{(x,\phi(x)a)}(b, \phi(x)c) + C_{(x,\phi(x)a)}(b, \phi'(x)(\lambda)a).
\]

Taking \(\phi\) constant, we see (7) implies

\[
C_{(x,a)} \text{ is homogeneous of degree } -1 \text{ in } a.
\]

Setting \(\phi(x) = \exp(-f(x))\), where \(f\) is a continuous linear functional on \(B\) such that \(f(\lambda) = 1\) (Hahn-Banach Theorem), we get \(C_{(x,a)}(\cdot, a) = 0\), i.e.

\[
3e(-, e) = 0,
\]

where we identify \((\mathcal{P}_0^{-1}E)_e\) with \(E_{pe}\). As a summary, we have

**Proposition 1.** A linear connection on \(\mathcal{P}_0^{-1}E \rightarrow E_0\) is a direction connection iff all its local components \((\Gamma, C)\) satisfy (6) and (7). Moreover, a direction connection satisfies (8) and (9).

**Corollary.** A linear connection on \(\mathcal{P}_0^{-1}E \rightarrow E_0\) with torsion zero is a direction connection.
Remark. In one of his earlier papers on Finsler geometry, M. Matsumoto [4] studied a more general class of linear connections on \( \mathcal{P}^{-1}E \rightarrow E_0 \), namely, those invariant under diffeomorphisms of \( \mathcal{P}^{-1}E \) defined by uniform radial expansions in the fibres of \( E_0 \rightarrow M \), i.e. (4) holds only for constant \( f \). Locally these connections are characterized by (6) and (8) (with only constant \( \phi \), (5) is equivalent to these two equations); (9) need not be satisfied. Let us call these connections \textit{weak direction connections} (in [4] they were called Finsler connections). Note that they differ from ordinary direction connections only by the behavior of the torsion tensor.

4. From \( \mathcal{D} \) to \( D \). Next we turn to the relationship between linear connections on \( \mathcal{P}^{-1}E \rightarrow E_0 \) and homogeneous nonlinear connections on \( \mathcal{P} : E \rightarrow M \).

In this section we derive a necessary and sufficient condition that a linear connection \( \mathcal{D} \) on \( \mathcal{P}^{-1}E \rightarrow E_0 \) defines a nonlinear connection \( D \) on \( \mathcal{P} : E \rightarrow M \) according to the prescription in [1, §5].

The vector bundle \( \mathcal{P}^{-1}E \rightarrow E \) has a canonical section \( \mathcal{V} : E_0 \rightarrow \mathcal{P}^{-1}E \) which is defined as \( \mathcal{V}(e) = e \) (using the identification \( \mathcal{P}^{-1}E \)) = \( E_{pe} \). Letting \( \mathcal{D} \) denote the connection map, define the map \( D : TE_0 \rightarrow E \) as \( D = r \circ \mathcal{D} \circ \mathcal{V} \). (This is just the covariant derivative map \( Z \rightarrow DZ \), see [5, §2].)

Proposition 2. For a linear connection \( \mathcal{D} \) on \( \mathcal{P}^{-1}E \rightarrow E_0 \), the map \( D = r \circ \mathcal{D} \circ \mathcal{V} \) defines a nonlinear connection on \( E \rightarrow M \) iff the torsion of \( \mathcal{D} \) satisfies

\[
\tau_{\mathcal{D}}(e, -) = 0.
\]

Moreover, if \( \mathcal{D} \) is a (weak) direction connection then the connection \( D \) is homogeneous.

Proof. We work locally. \( TE_0 \) is locally \( U \times E_0 \times B \times E \) and \( \mathcal{V}(x, a) = (x, a, a) \). Calculating \( \mathcal{V} \) and using (3) we get \( D(x, a, \lambda, b) = (x, b + \Gamma_{(x,a)}(a, \lambda) + C_{(x,a)}(a, b)) \). By [5, Lemma 1, p. 239] this defines a nonlinear connection iff \( \Gamma_{(x,a)}(a, \lambda) + C_{(x,a)}(a, b) \) is linear in \( \lambda \) and independent of \( b \). But this clearly occurs iff \( C_{(x,a)}(a, b) = 0 \) for all \( b \), which is precisely (10).

The local component of \( D \) is thus given by

\[
\omega(x, a) \lambda = \Gamma_{(x,a)}(a, \lambda).
\]

In case \( \mathcal{D} \) is a weak direction connection, \( \Gamma_{(x,a)} \) is homogeneous of degree 0 in \( a \). Since \( \Gamma_{(x,a)}(b, \lambda) \) is linear in \( b \), we see \( \omega(x, a) \) is homogeneous of degree 1 in \( a \), which means \( D \) is homogeneous. Q.E.D.
Let us say that in the setting of Proposition 2 the connection $\mathcal{D}$ is *associated* to the connection $D$. This means that (11) holds for local components.

**Remark.** Equation (10) is also known as E. Cartan's condition "D" (see [4]). This condition must be added to the results of Grifone in [3], since the step $\mathcal{D}$ to $D$ is used there. In Dazord [1] only direction connections with symmetric torsion ("regular") are treated, so that (9) implies (10).

5. From $D$ to $\mathcal{D}$. We now consider the reverse of the situation just discussed. Our results are as follows.

**Proposition 3.** For each homogeneous nonlinear connection $D$ on $E \to M$, there exists an associated direction connection $\mathcal{D}$ with torsion zero. If $D$ is linear, the pullback $r^{-1}D$ is such a $\mathcal{D}$.

$\mathcal{D}$ is not unique (even if it is assumed that dimensions are finite, $E = TM$, $D$ is linear and comes from a spray, and $\mathcal{D}$ is symmetric).

**Proof.** Using $Ve^*E = p^*E$ we apply the proposition of [6, §2], which assigns to $D$ the linear Berwald connection $\mathcal{D}$ on $p^*E \to E_0$. If $\omega$ denotes the local component of $D$, then that of $\mathcal{D}$ is by definition

$$\Gamma_{(x, a)}(b, \lambda) = \partial_\omega(x, a)(b)\lambda, \quad C_{(x, a)} = 0. \tag{12}$$

Hence clearly $\mathcal{D}$ has torsion zero. If $D$ is homogeneous, then $\omega$ is homogeneous of degree 1 in $a$, whence its derivative with respect to this variable is homogeneous of one less degree, namely 0. This means (6) holds for $\Gamma$. Since also (7) holds trivially for $C$, $\mathcal{D}$ is a direction connection by Proposition 1. Now the homogeneity of degree 1 of $\omega$ in $a$ implies by Euler's theorem that

$$\partial_\omega(x, a)(a)\lambda = \omega(x, a)\lambda. \tag{13}$$

But (12) and (13) give (11), which means $\mathcal{D}$ is associated to $D$. For $\omega$ linear in $a, \Gamma_{(x, a)}(b, \lambda) = \omega(x, b)\lambda$. This together with $C = 0$ means $\mathcal{D}$ is the pullback connection $r^{-1}D$.

To prove the nonuniqueness assertions let $M = R^3$ and $E \to M$ be the tangent bundle $R^2 \times R^2 \to R^2$ of $M$. Let $D$ be the linear connection defined by $\omega(x, a)\lambda = \langle a, \lambda \rangle x$, where $\langle \ , \ \rangle$ denotes the inner product in $R^2$. The Berwald connection $\mathcal{D}$ of $D$ is given by $C = 0$ and $\Gamma_{(x, a)}(b, \lambda) = \partial_\omega(x, a)(b)\lambda = \langle b, \lambda \rangle x$. Now define a direction connection $\mathcal{D}^0$ by $C^0 = 0$ and

$$\Gamma_{(x, a)}^0(b, \lambda) = [(a_1)^2b_1\lambda_1 + a_1a_2(b_1\lambda_2 + b_2\lambda_1) + (a_2)^2b_2\lambda_2](x/|a|^2), \tag{13}$$

where $a = (a_1, a_2)$. Then $\Gamma_{(x, a)}^0(a, \lambda) = \langle a, \lambda \rangle x = \omega(x, a)\lambda$, which means $\mathcal{D}^0$ is associated to $D$. But clearly $\mathcal{D} \neq \mathcal{D}^0$. 

Both $\mathcal{D}$ and $\mathcal{D}^0$ are symmetric, because it can be shown in a straightforward way from the definition of symmetry given in [1] that if torsion of $\mathcal{D}$ is zero, then $\mathcal{D}$ is symmetric iff the local component $\Gamma$ is symmetric.

To see that $D$ arises from a spray (see [1], [2]) on $M$, consider the spray defined by $G(x, a) = -(a, a)x$. Then with $\omega$ as above, we see that $\omega(x, a)\lambda = -\frac{1}{2}\partial_a G(x, a)(\lambda)$, which means $D$ comes from the spray $G$. Q.E.D.

**Remark.** Proposition 3 shows that the uniqueness statement in Theorem 1 [2] cannot be interpreted to mean one-to-one correspondence between symmetric torsion zero direction connections and their associated sprays. Therefore, it is not clear what sense this uniqueness statement could make.

If we add the assumption that $\mathcal{D}$ is continuously extendible to a connection on the bundle $p^{-1}E \to E$, and that $D$ is linear, then we do get uniqueness for torsion zero associated direction connections. For (as noted in [1]), the extendibility condition means locally that $\Gamma(x, a)$ does not depend on $a$. Differentiation of (11) then yields $\Gamma(x, a)(b, \lambda) = \omega(x, b)\lambda$, which together with $C = 0$ means that $\mathcal{D}$ is equal to the pullback connection $r^{-1}D$. (See also Matsumoto [4], where such connections are called simple connections.) However, the extendibility assumption is too restrictive for Finsler geometry.

**References**


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