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ON INTEGRABLE AND BOUNDED AUTOMORPHIC FORMS[1]

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ABSTRACT. A necessary and sufficient condition that every integrable automorphic form of dimension <-2 be a bounded form is established. Using this condition, it is shown that, for a finitely generated Fuchsian group acting on the unit disc and containing no parabolic elements, every integrable automorphic form of dimension <-2 is bounded. Here the dimension is not required to be integral. In the case of even integral dimension and standard factors of automorphy, this latter result is contained in D. Drasin and C. J. Earle, Proc. Amer. Math. Soc. 19 (1968), 1039–1042, but the present approach is entirely different. Also, using the argument of Drasin and Earle, it is proved that, for finitely generated Fuchsian groups of second kind, every integrable automorphic form of dimension -2 is zero.

1. **Introduction.** Throughout Γ denotes a Fuchsian group acting on the unit disc U of the complex plane. For any given real number q, we choose and fix, once and for all, a system $\rho(q, T, Z)$ of factors of automorphy belonging to Γ, i.e., a set of functions, one for each $T\in\Gamma$, defined for $Z\in U$ and satisfying

(i) $\rho(q, T, Z)$ is holomorphic in Z,

(ii) $|\rho(q, T, Z)| = |T'(Z)|^q$, and

(iii) $\rho(q, ST, Z)=\rho(q, S, TZ)\cdot\rho(q, T, Z)$, $\forall\, S, T\in\Gamma$ and $Z\in U$.

Note that, if q is an integer, $\rho(q, T, Z) =\chi(T)T'(Z)^q$, where χ is a character of Γ.

A function F holomorphic on U is said to be an automorphic form of dimension $-2q$ if $F(TZ)\rho(q, T, Z) = F(Z)$ $\forall\, T\in\Gamma$ and $Z\in U$. Let $\Omega\subset U$ be a fundamental region for Γ whose boundary has zero area. Following Bers [1], we denote by $A_q(\Gamma)$ the space of *integrable* forms, i.e., the set of all (holomorphic) automorphic forms F of dimension $-2q$ such that

$$(1.1)\qquad \|F\|_q \equiv \iint_\Omega |F(Z)|\,(1-|Z|^2)^{q-2}dxdy < \infty.$$

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$B_q(\Gamma)$ stands for the space of *bounded* forms, i.e., the set of all forms of dimension $-2q$ such that $F(Z)(1-|Z|^2)^q \in L_\infty(U)$. If $\Gamma=\{\mathrm{id}\}$ we write A_q and B_q for $A_q(\Gamma)$ and $B_q(\Gamma)$.

It is an open question whether, for an arbitrary Γ and $q>1$, $A_q(\Gamma)\subset B_q(\Gamma)$. Theorem 1 below provides a necessary and sufficient condition for this inclusion to hold. Theorem 2 of the paper shows that this condition is satisfied if Γ is finitely generated and does not contain parabolic transformations. For the case where q is an integer and $\rho(q, T, Z)=T'(Z)^q$, the inclusion $A_q(\Gamma)\subset B_q(\Gamma)$ was established by Drasin and Earle [**3**] by a judicious application of Abel's theorem on compact Riemann surfaces. We make no appeal to function theory on Riemann surfaces. We conclude the paper by pointing out that the method of Drasin and Earle [**3**] can be used to prove that $A_1(\Gamma)=\{0\}$ for certain groups.

2. **Results.** Before stating the results we need some more notation. For $q>1$, let $K(Z,\zeta)=\pi^{-1}(2q-1)(1-Z\bar{\zeta})^{-2q}$ where $K(0,\zeta)>0$. Let

$$\alpha(Z,\zeta)\equiv\alpha_q(Z,\zeta)\equiv\sum_{T\in\Gamma}\rho(q,T,Z)\cdot K(TZ,\zeta). \tag{2.1}$$

It is known that (every arrangement of) the Poincaré series defining α converges, for fixed $\zeta\in U$ and $q>1$, uniformly on compact subsets of U and that the function $Z\to\alpha(Z,\zeta)$ belongs to both $A_q(\Gamma)$ and $B_q(\Gamma)$ (cf. Earle [**4**, §4], and Drasin [**2**, Lemma 2]).

THEOREM 1. *Let* Γ *be arbitrary and* $q>1$. $A_q(\Gamma)\subset B_q(\Gamma)$ *if and only if*

$$\operatorname*{Sup}_{Z\in U}(1-|Z|^2)^{2q}\cdot\alpha(Z,Z)<\infty. \tag{2.2}$$

Note. It will be seen that $\alpha(Z,Z)\geqq 0$.

THEOREM 2. *Let* Γ *be finitely generated and contain no parabolic transformations. Then* (2·2) *holds and hence* $A_q(\Gamma)\subset B_q(\Gamma)$ *for* $q>1$.

THEOREM 3. *Let* Γ *be a finitely generated group of the second kind and* $\rho(1,T,Z)=T'(Z)$. *Then* $A_1(\Gamma)=\{0\}$.

3. **Proof of Theorem 1.** Let $F\in A_q(\Gamma)$. It is known (Drasin [**2**, equation (1.6)]; Earle [**4**, §4]) that for all ζ in U,

$$F(\zeta)=\iint_\Omega F(Z)\cdot\overline{\alpha(Z,\zeta)}(1-|Z|^2)^{2q-2}dxdy. \tag{3.1}$$

Hence

$$(3.2)\qquad \begin{aligned} &|F(\zeta)|(1-|\zeta|^2)^q \\ &\quad \leqq \|F\|_q\cdot \sup_{Z,\zeta\in U}(1-|Z|^2)^q(1-|\zeta|^2)^q\cdot|\alpha(Z,\zeta)|. \end{aligned}$$

Since $Z\rightarrow\alpha(Z,\zeta)\in A_q(\Gamma)$, we can conclude from (3.1) that

$$\alpha(\zeta,\zeta)=\iint_\Omega|\alpha(Z,\zeta)|^2(1-|Z|^2)^{2q-2}dxdy\geqq 0,$$

and

$$\alpha(Z,\zeta)=\iint_\Omega \alpha(w,\zeta)\cdot\overline{\alpha(w,Z)}(1-|w|^2)^{2q-2}dudv$$

$(w=u+iv)$. Combining these with Schwarz's inequality we infer that

$$(3.3)\qquad |\alpha(Z,\zeta)|^2\leqq\alpha(Z,Z)\cdot\alpha(\zeta,\zeta).$$

It follows from (3.2), (3.3) and (2.2) that $F\in B_q(\Gamma)$. That is to say, (2.2) is sufficient for the inclusion $A_q(\Gamma)\subset B_q(\Gamma)$ to hold.

Conversely, suppose that $A_q(\Gamma)\subset B_q(\Gamma)$. By the closed graph theorem, this inclusion is a bounded linear map from the Banach space $A_q(\Gamma)$ into the Banach space $B_q(\Gamma)$, i.e., there exists a positive constant $C\equiv C(\Gamma,q,\rho)$ such that, for all F in $A_q(\Gamma)$ and $Z\in U$, $|F(Z)|\cdot(1-|Z|^2)^q\leqq C\cdot\|F\|_q$. In particular, $|\alpha(Z,\zeta)|\cdot(1-|Z|^2)^q\leqq C\|\alpha(\cdot,\zeta)\|_q$, $Z,\zeta\in U$. Setting $Z=\zeta$ we obtain the inequality

$$(3.4)\qquad \alpha(\zeta,\zeta)(1-|\zeta|^2)^q\leqq C\cdot\|\alpha(\cdot,\zeta)\|_q.$$

Also (2.1) readily implies (cf. Bers [1, p. 202]) that

$$\begin{aligned}\|\alpha(\cdot,\zeta)\|_q&\leqq\iint_U|K(Z,\zeta)|\cdot(1-|Z|^2)^{q-2}dxdy\\ &=\pi^{-1}(2q-1)\cdot\iint_U\frac{(1-|Z|^2)^{q-2}}{|1-Z\bar\zeta|^{2q}}dxdy.\end{aligned}$$

Appealing to equation (2), p. 868, of Earle [**4**], we can now conclude that

$$\|\alpha(\cdot,\zeta)\|_q\leqq\frac{2q-1}{q-1}\cdot(1-|\zeta|^2)^{-q}.$$

Substituting this in (3.4) we infer that

$$\alpha(\zeta,\zeta)\cdot(1-|\zeta|^2)^{2q}\leqq C\cdot\frac{2q-1}{q-1},\qquad\forall\zeta\in U,$$

i.e. (2.2) holds. Q.E.D.

4. **Proof of Theorem 2.** We need only show that (2.2) holds. Let

$$G(Z) = (2q-1)\sum_{T\in\Gamma} |T'(Z)|^q \frac{(1-|Z|^2)^{2q}}{|1-\bar{Z}\cdot TZ|^{2q}}\cdot$$

It follows from (2.1) that

$$(4.1)\quad \pi(1-|Z|^2)^{2q}\cdot\alpha(Z,Z) \leqq G(Z) \leqq (2q-1)4^q \sum_{T\in\Gamma} |T'(Z)|^q.$$

The series $\sum_{T\in\Gamma}|T'(Z)|^q$ converges uniformly on compact subsets of $D\backslash\Gamma\infty$, where D is the *full* set of discontinuity of Γ in the extended complex plane. Since Γ is finitely generated without parabolic transformations, there exists a fundamental domain $\Omega\subset U$ such that the closure $\bar{\Omega}\subset D\backslash\Gamma\infty$ (Lehner [**5**, p. 145]). For such an Ω then, $\operatorname{Sup}_{Z\in\Omega}\sum_{T\in\Gamma}|T'(Z)|^q<\infty$ and, by (4.1),

$$(4.2)\qquad \operatorname*{Sup}_{Z\in\Omega}(1-|Z|^2)^{2q}\cdot\alpha(Z,Z) < \infty.$$

By the properties of the Bergman kernel for U,

$$\alpha(Z,\zeta) = \overline{\alpha(\zeta,Z)}$$

and it is then easily verified that $\beta(Z)\equiv(1-|Z|^2)^{2q}\alpha(Z,Z)$ is automorphic relative to Γ, i.e., $\beta\circ T=\beta$, $\forall T\in\Gamma$. This, together with (4.2), establishes (2.2). Q.E.D.

Remark. For the case where q is an integer and $\rho(q,T,Z)=T'(Z)^q$, Drasin and Earle [**3**] prove that $A_q(\Gamma)\subset B_q(\Gamma)$ for every finitely generated Γ, even if it contains parabolic elements. The simple estimate (4.1) is not good enough to handle the presence of parabolic elements. Indeed, a short computation shows that $G(Z)$ is unbounded on U if Γ contains a parabolic transformation. On the other hand, the method of Drasin and Earle [**3**] does not seem to be extendible to the case of nonintegral q. We hope to return to this matter.

5. **Proof of Theorem 3.** Let $\psi\in A_1(\Gamma)$. The proof of the lemma in Drasin and Earle [**3**] remains valid, word for word, for $q=1$ and we can conclude that $\psi=f\cdot\varphi$ where $f\in A_1$ and $\varphi\in A_1(\Gamma)$. Now $\lambda(t)\equiv\int_0^{2\pi}|f(te^{i\theta})|\,d\theta$ is an increasing function of t in $[0, 1)$. Hence, for any r in $[0, 1)$,

$$\infty > \|f\|_1 = \iint_U \frac{|f(Z)|}{1-|Z|^2}\,dxdy = \int_0^1 \frac{\lambda(t)tdt}{1-t^2} \geqq \lambda(r)\int_r^1 \frac{tdt}{1-t^2}$$
$$=\lambda(r)\cdot\infty.$$

It follows that $\lambda(r)=0$ for all r in $[0, 1)$. Hence $f(Z)\equiv 0$, so that $\psi(Z)\equiv 0$. Q.E.D.

Note. If Γ is finitely generated and is of the first kind, $A_1(\Gamma) = B_1(\Gamma) \neq \{0\}$.

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