ON THE CONVERGENCE OF MULTIPLICATIVELY ORTHOGONAL SERIES

C. J. PRESTON

Abstract. G. Alexits and A. Sharma have recently shown that if \( \{\varphi_n\}_{n=1}^{\infty} \) is a uniformly bounded multiplicatively orthogonal system on a finite measure space and if \( \{c_n\}_{n=1}^{\infty} \) is a sequence of real numbers with \( \sum_{n=1}^{\infty} c_n^2 < \infty \), then the partial sums \( \sum_{n=1}^{\infty} c_n \varphi_n \) converge almost everywhere. We give here a simple proof of this result.

Let \( (X, \mathcal{B}, \mu) \) be a measure space, with \( \mu \) a finite nonnegative measure, and let \( f_n : X \rightarrow \mathbb{R}, n = 1, 2, \ldots, \) be an orthonormal system on \( (X, \mathcal{B}, \mu) \), (i.e. \( f_n \in L^2(X, \mathcal{B}, \mu) \) with \( \int_X f_n f_m \, d\mu = \delta_{n,m} \)). Let \( c_n \in \mathbb{R}, n = 1, 2, \ldots, \), and define \( s_n(x) = \sum_{i=1}^{n} c_i \varphi_i(x) \). Then the classical result of Menchoff states that \( s_n \) converges a.e. as \( n \to \infty \), provided \( \sum_{n=1}^{\infty} c_n^2 (\log n)^2 < \infty \). Menchoff also showed that for a general orthonormal system this is the best result possible. For particular orthonormal systems we can get better results; for example, if \( X = T \), and \( \mu \) = Lebesgue measure on \( T \), and \( f_n(x) = \cos nx \), or \( f_n(x) = \sin nx \), then it follows from the famous result of Carleson that \( s_n \) converges a.e. as \( n \to \infty \) provided \( \sum_{n=1}^{\infty} c_n^2 < \infty \). In a preprint of a paper to appear in Acta Math. Acad. Sci. Hungar., G. Alexits and A. Sharma prove a similar result for uniformly bounded multiplicatively orthogonal systems. (We say \( \{\varphi_n\}_{n=1}^{\infty} \) is a uniformly bounded multiplicatively orthogonal system on \( (X, \mathcal{B}, \mu) \) if \( \varphi_n \in L^\infty(X, \mathcal{B}, \mu) \) with \( \| \varphi_n \|_\infty \leq M \) for some \( M \) and all \( n \), and if given any \( m = 1, 2, \ldots, \), and

\[ 1 \leq n_1 < n_2 < \cdots < n_m, \text{ then } \int_X \varphi_{n_1} \cdots \varphi_{n_m} \, d\mu = 0. \]

Alexits and Sharma prove the following:

**Theorem.** Let \( \{\varphi_n\}_{n=1}^{\infty} \) be a uniformly bounded multiplicatively orthogonal system on \( (X, \mathcal{B}, \mu) \). Let \( c_n \in \mathbb{R}, n = 1, 2, \ldots, \), and let \( s_n(x) = \sum_{n=1}^{\infty} c_n \varphi_n(x) \). Then \( s_n \) converges a.e. as \( n \to \infty \) provided \( \sum_{n=1}^{\infty} c_n^2 < \infty \).

The proof of this theorem by Alexits and Sharma involves some difficult constructions; we give here a short and simple proof.

We may suppose without loss of generality that \( |\varphi_n(x)| \leq 1 \) for all \( x \in X \) and for all \( n \). Let \( \{\psi_n\}_{n=0}^{\infty} \) be the product system associated with \( \{\varphi_n\}_{n=1}^{\infty} \); i.e.

Received by the editors July 27, 1970.

AMS 1970 subject classifications. Primary 40A05, 42A60, 42A20.

Key words and phrases. Uniformly bounded multiplicatively orthogonal systems, almost everywhere convergence.
\[ \psi_n = \varphi_{r+1} \cdots \varphi_{m+1} \text{ for } n = 2^r + \cdots + 2^m, \]
\[ \psi_0 = 1. \]

Note the following two facts:

1. \[ \int_X \psi_n \, d\mu = 0, \quad n = 1, 2, \ldots; \]
2. \[ \sum_{k=0}^{2^m-1} \psi_k(x)\psi_k(y) = \prod_{k=1}^{m} (1 + \varphi_k(x)\varphi_k(y)) \geq 0 \text{ for all } x, y \in X. \]

Define \( n(x) \) to be the least index such that \( s_{n(x)}(x) = \max_{1 \leq s \leq n} s_r(x) \). We have
\[ s_n(x) = \sum_{k=0}^{2^n-1} a_k \psi_k(x), \]
where
\[ a_k = c_{r+1} \text{ if } k = 2^r, \]
\[ = 0 \text{ otherwise}, \]
and so \( s_{n(x)}(x) = \sum_{k=0}^{2^n(x)-1} a_k \psi_k(x) \).

Let \((Y, \alpha, \omega)\) be any finite measure space, and let \( \{g_n\}_{n=0}^{\infty} \) be any orthonormal system on \((Y, \alpha, \omega)\). Then
\[ s_{n(x)}(x) = \int_X \sum_{k=0}^{2^n-1} a_k g_k(t) \sum_{j=0}^{2^n(x)-1} \psi_j(x)g_j(t) \, d\omega(t). \]

Therefore
\[ \left| \int_X s_{n(x)}(x) \, d\mu(x) \right| \leq \left( \int_Y \left[ \sum_{k=0}^{2^n-1} a_k g_k(t) \right]^2 \, d\omega(t) \right)^{1/2} \]
\[ \cdot \left( \int_X \left[ \sum_{k=0}^{2^n(x)-1} \psi_k(x)g_k(t) \, d\mu(x) \right]^2 \, d\omega(t) \right)^{1/2} \]
\[ = \left( \sum_{k=0}^{2^n-1} a_k^2 \right)^{1/2} \left( \sum_{j=0}^{2^n(x)-1} \psi_j(y)g_j(t) \, d\mu(x) \, d\mu(y) \, d\omega(t) \right)^{1/2}. \]
Thus
\[
\left| \int_X s_{n(x)}(x) \, d\mu(x) \right|^2
\]
\[
\leq \left( \sum_{k=1}^{n} c_k^2 \right) \int_X \int_X \int_Y \sum_{k=0}^{2^n(x)-1} \psi_k(x) g_k(t) \sum_{j=0}^{2^n(y)-1} \psi_j(y) g_j(t) \, d\omega(t) \, d\mu(x) \, d\mu(y)
\]
\[
= \left( \sum_{k=1}^{n} c_k^2 \right) \int_X \int_X \sum_{k=0}^{2^n(x)-1} \psi_k(x) \psi_k(y) \, d\mu(x) \, d\mu(y),
\]
where \( n(x, y) = \min\{n(x), n(y)\} \).

\[
\leq 2 \left( \sum_{k=1}^{n} c_k^2 \right) \int_X \int_X \left| \sum_{k=0}^{2^n(x)-1} \psi_k(x) \psi_k(y) \right| \, d\mu(x) \, d\mu(y)
\]
\[
= 2 \left( \sum_{k=1}^{n} c_k^2 \right) \int_X \int_X \sum_{k=0}^{2^n(x)-1} \psi_k(x) \psi_k(y) \, d\mu(x) \, d\mu(y) \quad \text{(using (2))}
\]
\[
= 2 \left( \sum_{k=1}^{n} c_k^2 \right) \int_X \int_X \psi_0(x) \psi_0(y) \, d\mu(x) \, d\mu(y) \quad \text{(using (1))}
\]
\[
= 2 \left( \sum_{k=1}^{n} c_k^2 \right) [\mu(X)]^2.
\]

Hence we have
\[
\left| \int_X s_{n(x)}(x) \, d\mu(x) \right|^2 \leq 2 \left( \sum_{k=1}^{n} c_k^2 \right) [\mu(X)]^2.
\]

It is well known that such an estimate is sufficient in order to prove the theorem.

Cornell University, Ithaca, New York 14850