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# A CLASS OF HYPO-DIRICHLET ALGEBRAS

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ABSTRACT. A method is given of constructing a new class of hypo-Dirichlet algebras of given real codimension.

1. **Introduction.** Let $X$ be a compact Hausdorff space and $A$ a uniform algebra on $X$, i.e., a uniformly closed subalgebra of $C(X)$, the space of continuous functions on $X$, that contains the constants and separates points on $X$. Denote the real parts of the functions in $A$ by $\operatorname{Re} A$, the set of invertible elements of $A$ by $A^{-1}$, the set of logarithms of moduli of functions in $A$ by $\log A$. Let $C(X)$ denote the space of real continuous functions on $X$. A uniform algebra on $X$ is called a *hypo-Dirichlet* algebra if, in addition, there exist $f_1, \cdots, f_n$ in $A^{-1}$, such that the (real) vector space spanned by $\operatorname{Re} A$ and $\log |f_1|, \cdots, \log |f_n|$ is dense in $C(X)$. The minimal number of such '$f_i$' required shall be called the *codimension* of $\operatorname{Re} A$. Hypo-Dirichlet algebras were first studied by Wermer [6], and further investigated by Ahern and Sarason [1]. The object here is to exhibit a class of examples of such algebras. The proofs of several of the lemmas in this paper are modeled after [2].

2. **The algebra** $A$. Let $\Gamma$ be the annulus $\{Z:1\leqq |Z| \leqq 2\}$, $\gamma_1 = \{Z:|Z|=1\}$ and $\gamma_2 = \{Z:|Z|=2\}$. Let $\Psi$ be a homeomorphism of $\gamma_1$ on $\gamma_2$ which is orientation-preserving and singular, i.e., maps a Borel set of one-dimensional Lebesgue measure 0 onto a set of measure $4\pi$. Let $B=\{f\in C(\Gamma):f$ is analytic in $\operatorname{int}(\Gamma)\}$, and $A =\{f\in B:f(Z)=f(\Psi(Z))\}$ for all $Z\in\gamma_1$. Let $A_\Psi = A$ restricted to $\gamma_1$. Then $A_\Psi$ is a uniformly closed algebra of continuous functions on $\gamma_1$, which contains the constants.

THEOREM. *$A_\Psi$ is a hypo-Dirichlet algebra on $\gamma_1$, and* $\operatorname{Re} A_\Psi$ *has codimension* 1 *in* $C_R(\gamma_1)$.

DEFINITION 1. A (complex Borel) measure $\nu$ on $\gamma_1\cup\gamma_2$ is odd if for each Borel set $E\subset\gamma_1$, $\nu(E) = -\nu(\Psi(E))$.

DEFINITION 2. $H$ denotes the class of measures of the form: $g(Z)\,dZ$ on $\gamma_1\cup\gamma_2$, where $g$ is any function in the $L^1$ closure of $B$ restricted to $\gamma_1\cup\gamma_2$.

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DEFINITION 3. $W$ is the space of measures $\mu+\nu$ with $\mu\in H$, $\nu$ odd. $\overline{W}$ is the weak * closure of $W$ in the space of measures on $\gamma_1\cup\gamma_2$.

DEFINITION 4. A measure $\lambda$ on $\gamma_1\cup\gamma_2$ *annihilates* $A$ if $\int f\,d\lambda=0$, all $f\in A$.

Clearly, every measure in $W$ annihilates $A$. Also, if $\lambda$ annihilates $A$, then $\lambda\in\overline{W}$.

NOTE. The measure $-i\cdot(dZ/Z)=d\theta$ is a real measure which annihilates $B$, and it is readily seen that the only real annihilators of $B$ are of the form: $\alpha\cdot d\theta$, $\alpha$ a real constant.

LEMMA 1. *If $\mu\in H$, $\nu$ odd, then* $\|\nu\|\leqq 16\|\mu+\nu\|$.

PROOF. Let $E$ be any Borel subset of $\gamma_1$ and let $m$ represent Lebesgue measure. Then there are disjoint sets $F$ and $G$ with $E=F\cup G$, $m(F)=m(\Psi(G))=0$. Let $K=\|\mu+\nu\|$, then $|\nu(F)|=|\nu(F)+\mu(F)|\leqq K$, since $\mu$ is absolutely continuous. $|\nu(G)|=|\nu(\Psi(G))|=|(\mu+\nu)(\Psi(G))|\leqq K$ for the same reason. Hence $\|\nu\|\leqq 16\,K$. q.e.d.

LEMMA 2. *Then* $W=\overline{W}$.

PROOF. $Q=\{\mu+\nu:\mu\in H,\ \nu \text{ odd},\ \|\mu\|\leqq 1,\ \|\nu\|\leqq 1\}$ is compact. The Krein-Smulian theorem [4, p. 429] then implies $W=\overline{W}$. q.e.d.

LEMMA 3. *If $\nu$ is an odd measure, then $\nu$ is absolutely continuous with respect to arc length on $\gamma_1\cup\gamma_2$ iff $\nu=0$.*

PROOF. Suppose $\nu$ is absolutely continuous. Let $E$ be a Borel subset of $\gamma_1$. Then there are disjoint sets $F$ and $G$ with $E=F\cup G$, $m(F)=m(\Psi(G))=0$. Hence $\nu(F)=0$, since $\nu$ is absolutely continuous $\nu(G)=-\nu(\Psi(G))=0$ for the same reason. Hence $\nu(E)=0$. q.e.d.

LEMMA 4. *Every real annihilator, $\lambda$, of $A$ has the form: $\lambda=\nu+\alpha\cdot d\theta$, where $\nu$ is odd and $\alpha$ is a real scalar.*

PROOF. Since $W$ is weak * closed, we conclude that if $\lambda$ is a measure on $\gamma_1\cup\gamma_2$, which annihilates $A$, then $\lambda=\mu+\nu$, $\mu\in H$, $\nu$ odd. Write $\mu=\mu_1+i\mu_2$, $\nu=\nu_1+i\nu_2$ with $\mu_1$, $\mu_2$, $\nu_1$, and $\nu_2$ real. If $\lambda$ is real $\mu_2+\nu_2=0$. Hence $\nu_2$ is absolutely continuous, hence 0. Then $\mu=\mu_1$ and $\lambda=\mu_1+\nu_1$. q.e.d.

It follows, in particular, that Re $A$ has codimension $\leqq 1$ in $C_R(\gamma_1)$.

LEMMA 5. *$A$ separates the points of $\gamma_1$. Further, given $Z_1$, $Z_2$ with $1\leqq|Z_1|\leqq 2$, $1\leqq|Z_2|<2$ and $Z_1\neq Z_2$, then there exists an $f$ in $A$ such that $f(Z_1)\neq f(Z_2)$.*

PROOF. Let $\tau_1$, $\tau_2$ be two points of $\gamma_1$ and let $\delta_{\tau_1}$, $\delta_{\tau_2}$ be the point masses at $\tau_1$, $\tau_2$ respectively. Unless $A$ separates $\tau_1$ and $\tau_2$, $\delta_{\tau_1}-\delta_{\tau_2}$ will be a real annihilating measure which is not in $W$. Now suppose $Z_1$, $Z_2$ are interior to the annulus and $A$ fails to separate them. Let $\sigma_1$, $\sigma_2$ be the harmonic measures for $Z_1$, $Z_2$ respectively. Then $\sigma_1-\sigma_2$ would be a real annihilating measure. Hence, $\sigma_1-\sigma_2=\nu+\alpha\cdot d\theta$, $\nu$ odd. However $\sigma_1-\sigma_2$ is absolutely continuous, therefore $\sigma_1-\sigma_2=\alpha\cdot d\theta$, contradiction. Finally, if $Z_1\in\gamma_1$ and $Z_2$ is interior, a similar argument applies. q.e.d.

Let $T$ be the space obtained from the closed annulus $1\leqq|Z|\leqq 2$ by by identifying $Z$ and $\Psi(Z)$ if $Z\in\gamma_1$. Then functions in $A$ may be regarded as continuous functions on $T$. Evidently $T$ is topologically a torus since $\Psi$ is orientation-preserving.

LEMMA 6. *The space of maximal ideals of $A$ $(A_\Psi)$ is homeomorphic to $T$.*

PROOF. It must be shown that, if $h$ is a homomorphism of $A$ onto the complex numbers, then $h$ is evaluation at some point of $T$. If $h$ is not evaluation at any point of $T$, then for each $Z$, $1\leqq|Z|\leqq 2$, there is an $f_Z\in A$, with $h(f_Z)=0, f_Z(Z)\neq 0$. Since $T$ is compact, we can select a finite number of functions $f_1,\cdots,f_n$ in $A$ such that $h(f_i)=0$ and open sets $\Delta_i$ in $1\leqq|Z|\leqq 2$ such that $\bigcup_1^n\Delta_i=\{Z:1\leqq|Z|\leqq 2\}$ and $f_i\neq 0$ in $\Delta_i$. Let $\sigma$ be a representing measure for $h$ on $\gamma_1\cup\gamma_2$, i.e., $h(f)=\int f\,d\sigma$, all $f\in A$. Then $\int f\cdot f_i\,d\sigma=h(f\cdot f_i)=h(f)\cdot h(f_i)=0, i=1,\cdots,n$, $f\in A$. Thus $f_i\cdot d\sigma$ annihilates $A$, therefore $f_i\cdot d\sigma=d\mu_i+d\nu_i$, $\mu_i\in H$, $\nu_i$ odd. Hence, $f_j(d\mu_i+d\nu_i)=f_j\cdot f_i\,d\sigma=f_i\cdot(d\mu_j+d\nu_j)$ and so $f_j\cdot d\mu_i-f_i d\mu_j=f_i\cdot d\nu_j-f_j\cdot d\nu_i$. Since the right side is odd and the left side is absolutely continuous both sides vanish. Let $\Phi_i$ denote the function in $H$ such that $d\mu_i=\Phi_i\cdot dZ$. Then $f_j\cdot\Phi_i=f_i\cdot\Phi_j$ a.e. on $\gamma_1\cup\gamma_2$ and so $f_j\cdot\Phi_i=f_i\cdot\Phi_j$ also for $1<|Z|<2$. We can therefore unambiguously define $\Phi$ on $1\leqq|Z|\leqq 2$ by $\Phi(z)=\Phi_i(Z)\cdot(f_i(Z))^{-1}$ for $Z\in\Delta_i$. Then $\Phi\in H$.

We define a measure $\nu$ on $\gamma_1\cup\gamma_2$ by $d\nu=(f_i)^{-1}\cdot d\nu_i$ on $(\gamma_1\cup\gamma_2)\cap\Delta_i$. Then $\nu$ is well defined and odd. Then $f_i\,d\sigma=f_i\cdot\Phi\cdot dZ+f_i\,d\nu$. Since $f_i\neq 0$ on $\Delta_i$, we deduce $d\sigma=\Phi dZ+d\nu$. But then $1=\int d\sigma=\int\Phi\cdot dZ+\int d\nu=0$. Contradiction. q.e.d.

LEMMA 7. *There is an $f\in A^{-1}$ whose logarithm is not single valued on $\Gamma$.*

PROOF. We regard $A$ as an algebra of continuous functions on $T$. The circle: $|Z|=3/2$ gives rise to a one-cycle $l_1$ on $T$. Let $l_2$ be another one-cycle on $T$ so that $l_1$ and $l_2$ generate $H_1(T, Z)$. By a theorem of Arens-Royden, [5], the quotient group $A^{-1}/\exp(A)$ is isomorphic to $H^1(T, Z)$. If $T$ is the torus, $H^1(T, Z)$ is a free abelian group on two generators. Let $g_1$, $g_2$ be two elements of $A^{-1}$ representing these generators. Write $g_1=e^{h_1}$, $g_2=e^{h_2}$, where $h_1$, $h_2$ are (multi-valued) analytic functions on $\Gamma$. Let $h_1$ have period $2n\pi i$ on $l_2$, $h_2$ have period $2m\pi i$ on $l_2$. Then $m\cdot h_1-n\cdot h_2$ has 0 period on $l_2$. Suppose it also had 0 period on $l_1$. Then $g_1^m\cdot g^{-n}=e^h$ for some $h\in A$. This contradicts the choice of $g_1$, $g_2$. Hence $mh-nh$ has period $\neq 0$ on $l$. Therefore $f=g_1^m\cdot g_2^{-n}$ is the desired element of $A^{-1}$. q.e.d.

Proof of Theorem. We must show that there is an $f\in A_\psi^{-1}$ such that $\log|f|\notin$ closure Re $A$. We claim the $f$ of Lemma 7 is such a function. Define a linear functional $L$ on $C_R(\gamma_1\cup\gamma_2)$ by $L(U)=(1/2\pi)\int_{|Z|=3/2}dv$ where $v$ is the harmonic conjugate of $U$. Then $L$ is continuous and linear. $L(g)=0$ for $g\in\mathrm{Re}\,A^{-1}$, but $L(\log|f|)\neq 0$ since $\int_{|Z|=3/2}(\arg f)\neq 0$. Therefore $\log|f|\notin$ closure Re $A^{-1}$. q.e.d.

Note. By identifying $n$ circles instead of 2, in a similar manner, we can construct a hypo-Dirichlet algebra that has real part of codimension $n-1$.

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